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METHODES NOUVELLES DE LA MECANIQUE CELESTE

(H. Poincaré)

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Paul Appell déclarait en 1925, à propos des *Méthodes Nouvelles de la Mécanique Céleste* : "Il est probable que, pendant le prochain demi-siècle, ce livre sera la mine d'où des chercheurs plus humbles extrairont leurs matériaux".

Aujourd'hui, près de cent ans après la parution des *Méthodes Nouvelles,* cette affirmation est toujours vérifiée. Cependant, l'éclatement de son contenu en différents domaines (Mécanique Céleste, Systèmes Dynamiques, Géométrie Différentielle,...) rend difficile la lecture de ce texte fondamental. Nombreux sont ceux qui l'ont cité, certains en ont étudié différents chapitres, mais rare sont ceux qui connaissent l'intégralité de ses trois volumes.

C'est pourquoi nous avons entrepris, à travers ce groupe de travail qui rassemble mathématiciens et astronomes, une lecture collective des *Méthodes Nouvelles*.

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THE METHOD OF DELAUNAY

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1 The method of Delaunay's theory of the Moon

Delaunay has been the first astronomer to use the Mechanics of Hamilton to obtain the approximated solutions of the equations of the motion of a celestial body. His theory of the Moon (see Charlier, 1902) is a pioneer work in many respects. We credit to Delaunay the introduction of a set of variables – the so-called Delaunay variables L, G, H, ℓ, g, h – in which Lagrange's equations for the variation of the orbital elements under a perturbation are canonical. His theory of the Moon is not a collection of tricks, as almost every higher-order theory in classical Celestial Mechanics. Generally speaking, having obtained the variation equations in canonical form, his problem was to find the solutions of the differential equations defined by the Hamiltonian

$$F = F_0(x_i) + \varepsilon \sum_k A_k(x_i) \cos(k \mid \theta), \tag{1}$$

where the canonical variables are (x_i, θ_i) $(i = 1, ..., n)$. ε is a small parameter and the sum is extended to all vectors $k \in \mathbf{D} \subset \mathbf{Z}^n$. The technique adopted by Delaunay is methodologically very clear. He defined an *operation* and performed it, successively, almost 500 times. This operation starts with the choice of one argument $(k_1 \mid \theta)$ in eqn. 1 and the consideration of the dynamical system defined by the abridged Hamiltonian

$$F_1 = F_0(x_i) + \varepsilon A_{k_1}(x_i) \cos(k_1 \mid \theta). \tag{2}$$

This system is integrable, since the angles θ_i are present only through the linear combination $(k_1 \mid \theta)$. The main step of one Delaunay's operation is to obtain a particular solution of the abridged problem and to use this solution to derive a canonical transformation leading to the elimination of the term $\varepsilon A_{k_1}(x_i) \cos(k_1 \mid \theta)$ from the given Hamiltonian (in fact the substitution of this term by others with much smaller coefficients).

To obtain the solution of the dynamical system defined by F_1, we introduce the Jacobian generating function

$$S(x^*, \theta) = (x^* \mid \theta) + S_1(x^*, \theta) \tag{3}$$

and consider the Hamilton-Jacobi equation

$$F_1^*(x^*) = F_0(\frac{\partial S}{\partial \theta_i}) + \varepsilon A_{k_1}(\frac{\partial S}{\partial \theta_i})\cos(k_1 \mid \theta) \tag{4}$$

The functions of $(\frac{\partial S}{\partial \theta_i})$, in the right-hand side of this equation, may be expanded about $\frac{\partial S}{\partial \theta_i} = x_i^*$ and there results

$$F_1^*(x^*) = F_0(x_i^*) + \sum_i \frac{\partial F_0}{\partial x_i^*}\frac{\partial S_1}{\partial \theta_i} + \cdots + \varepsilon A_{k_1}(x_i^*)\cos(k_1 \mid \theta) + \cdots \tag{5}$$

The non-written terms are of the order $\mathcal{O}(\varepsilon^2)$ (since S_1 is of order $\mathcal{O}(\varepsilon)$, as shown thereafter).

An approximated particular solution of this Hamilton-Jacobi equation is obtained by choosing S_1 to be such that

$$\sum_i \frac{\partial F_0}{\partial x_i^*}\frac{\partial S_1}{\partial \theta_i} + \varepsilon A_{k_1}(x_i^*)\cos(k_1 \mid \theta) = 0, \tag{6}$$

that is,

$$S_1 = -\frac{\varepsilon A_{k_1}(x_i^*)\sin(k_1 \mid \theta)}{(k_1 \mid n)}, \tag{7}$$

where n is the vector $n = (n_1, ..., n_n)$ and

$$n_i = \frac{\partial F_0}{\partial x_i^*}. \tag{8}$$

Once solved the system defined by F_1, Delaunay goes back to the given Hamiltonian F and performs the transformation of the variables generated by the function S:

$$x_i = \frac{\partial S}{\partial \theta_i} = x_i^* + \frac{\partial S_1}{\partial \theta_i} \qquad \theta_i^* = \frac{\partial S}{\partial x_i^*} = \theta_i + \frac{\partial S_1}{\partial x_i^*}. \tag{9}$$

To complete the exposition of the technique of Delaunay let us write

$$F = F_1 + \Delta F. \tag{10}$$

Hence, according to eqns. 5 and 6, when F_1 is transformed, the term whose argument is $(k_1 \mid \theta)$ disappears:

$$F_1^*(x^*) = F_0(x_i^*) + \mathcal{O}(\varepsilon^2). \tag{11}$$

The additional part $\Delta F(x, \theta)$ is transformed into $\Delta F(x^*, \theta^*) + \mathcal{O}(\varepsilon^2)$ (the function ΔF being the same as before).

The result of the Delaunay operation is then a new Hamiltonian, differing, formally, from the given one, in only two respects:

- The term $\varepsilon A_{k_1} \cos(k_1 \mid \theta)$ disappears from F;
- Terms of order $\mathcal{O}(\varepsilon^2)$ are added to F.

In this way, performing as many operations as necessary, we may expect to eliminate from F all periodic terms of order $\mathcal{O}(\varepsilon)$. In fact all these operations can be performed together, finding just one function S which eliminates all periodic terms of order $\mathcal{O}(\varepsilon)$. This more direct method was discussed by Poincaré under the title *Lindstedt's method* and gave origin, later, to the method of Von Zeipel, widely used in the past 30 years(see Ferraz-Mello, 1989).

We may also expect to eliminate, with a second sequence of operations, those terms of order $\mathcal{O}(\varepsilon^2)$; after that, the terms of order $\mathcal{O}(\varepsilon^3)$, and so on.

In fact this is not so. Poincaré has shown that, as the work progresses, the combination of the arguments $(k \mid \theta)$ in the transformation of ΔF tends to enlarge the set of values of k (the maximum of $\mid k \mid$ increases) reaching some values of k for which $(k \mid n)$ is less than any given limit. Thus, the method of Delaunay may not be extended indefinitely and only a finite number of operations may be done.

We may also consider the case where one or more of the values $k \in \mathbf{D}$ are already such that $(k \mid n) = 0$. This case happens, for instance, when $F_0(x)$ is degenerated, that is, when F_0 does not depend on all components of x. This kind of degeneracy is called *essential*, as it is independent of the initial conditions. This case is general in Celestial Mechanics where F_0 depends only on the Delaunay variable L and on the variable Λ canonically conjugate to the time t:

$$F_0 = \frac{\mu^2}{2L^2} + \Lambda. \tag{12}$$

In this case the method of Delaunay does not allow us to eliminate, from the disturbing function

$$\sum_k \varepsilon A_k \cos(k \mid \theta), \tag{13}$$

those periodic terms independent of both, the time t and the mean anomaly ℓ (conjugate to L). In the particular problem of the motion of the Moon, periodic terms of this kind do not exist in the Hamiltonian and the method of Delaunay allows us to eliminate all periodic terms of order $\mathcal{O}(\varepsilon)$.

In the next orders, instead of F_0, we may use the non-degenerate Hamiltonian

$$F_0' = F_0(L, \Lambda) + \varepsilon A_0(L, G, H) \tag{14}$$

where εA_0 is the secular term of order $\mathcal{O}(\varepsilon)$ (i.e. the first term of the disturbing function corresponding to $k = 0$).

The condition $(k \mid n) = 0$ may also be verified for some $k \in \mathbf{D}$, or for some k generated in the calculations, but for a particular set of initial conditions. In this case we have what is known as a *resonance* or an *accidental* degeneracy. It will be considered in the next section.

2 The Method of Delaunay according to Poincaré

Poincaré (1893, chap.XIX) considered the method of Delaunay in the first part of his chapter on the method of Bohlin and restricted his study to the case where a resonance exists. Poincaré considered that

$$(k_1 \mid n) = 0 \tag{15}$$

for some $k_1 \in \mathbf{D}$ and for some point $(x_1, ..., x_n)$ of the domain of the x-space under study.

We may proceed as before. However, because of eqn. 15, we write also the quadratic term in the expansion of $F_0(\frac{\partial S}{\partial x_i})$. Then, instead of eqn. 5, we write

$$F^*(x^*) = F_0(x_i^*) + \sum_i \frac{\partial F_0}{\partial x_i^*}\frac{\partial S_1}{\partial \theta_i} + \frac{1}{2}\sum_i\sum_j \frac{\partial^2 F_0}{\partial x_i^* \partial x_j^*}\frac{\partial S_1}{\partial \theta_i}\frac{\partial S_1}{\partial \theta_j} + \cdots$$

$$+ \sum_k \varepsilon A_k(x_i^*)\cos(k \mid \theta) + \cdots \tag{16}$$

In fact, because of the difficulties introduced by the resonance, Poincaré considered only the one-degree-of-freedom case where the disturbing function is formed just by one term. Without loss of generality, we choose this term to be $\varepsilon A_1 \cos\theta_1$ and write

$$F^*(x_1^*) = F_0(x_1^*) + n_1(\frac{\partial S_1}{\partial \theta_1}) + \frac{1}{2}m_1(\frac{\partial S_1}{\partial \theta_1})^2 + \cdots + \varepsilon A_1(x_1^*)\cos\theta_1 + \cdots, \tag{17}$$

where we introduced

$$n_1 = \frac{dF_0}{dx_1^*}; \qquad m_1 = \frac{d^2F_0}{dx_1^{*2}}. \tag{18}$$

As before, an approximated solution of the Hamilton-Jacobi equation is obtained by choosing S_1 to be such that

$$n_1(\frac{\partial S_1}{\partial \theta_1}) + \frac{1}{2}m_1(\frac{\partial S_1}{\partial \theta_1})^2 + \varepsilon A_1(x_1^*)\cos\theta_1 = C. \tag{19}$$

The difference between this equation and eqn. 6 are the right- hand side C – necessary for having a complete solution – and the presence of the quadratic term, necessary because n_1 may become zero in the domain under study.

We may solve eqn. (19) to obtain

$$\frac{\partial S_1}{\partial \theta_1} = -\frac{n_1}{m_1} \pm \frac{1}{m_1}\sqrt{n_1^2 + 2m_1(C - \varepsilon A_1 \cos\theta_1)}. \tag{20}$$

The inclusion of n_1 in the right-hand side rises some difficulties. As it was clearly pointed out by Garfinkel, Jupp and Williams (1971), Poincaré follows Bohlin and assumes that n_1 is a small quantity of the order $\mathcal{O}(\sqrt{\varepsilon})$. One then faces the fact that differentiations may change the order of a term, since

$$\frac{dn_1}{dx_1^*} = m_1$$

is finite.

These difficulties are, in a certain way, addressed in §201 of Poincaré's book. This paragraph was thoroughly studied by Jupp(1973) who showed that some results given by Poincaré cannot be reproduced, since Poincaré, apparently, discarded some important terms in his equation for the derivative of the function T_1 of page 337. On the other hand, Poincaré considered, separately, the case $n_1 = 0$ (§199) and the general case $n_1 \neq 0$ (but close to zero) (§§199-200). This separation is not necessary, as shown thereafter where the derivations of these paragraphs are given, but with an interpretation diverse of Poincaré's one.

Let us fix the value of x_1^* to be such that $n_1 = 0$, exactly, and let us assume that the unknown function $S = x_1^*\theta_1 + S_1(C_1, \theta_1)$ is the Jacobian generator of one canonical transformation

$$(x_1, \theta_1) \Rightarrow (C, \gamma),$$

that is, the transformation

$$x_1 = \frac{\partial S}{\partial \theta_1} = x_1^* + \frac{\partial S_1}{\partial \theta_1}$$
$$\gamma = \frac{\partial S}{\partial C} = \frac{\partial S_1}{\partial C}, \tag{21}$$

instead of that given by eqns. 9. Equation 19 still holds but, as x_1^* was fixed to be such that $n_1(x_1^*) = 0$, it is reduced to

$$\frac{1}{2}m_1(\frac{\partial S_1}{\partial \theta_1})^2 + \varepsilon A_k(x_1^*)\cos\theta_1 = C \tag{22}$$

or

$$\frac{\partial S_1}{\partial \theta_1} = \sqrt{\frac{2}{m_1}(C - \varepsilon A_1 \cos\theta_1)}, \tag{23}$$

where we assumed $m_1 > 0$.

Equation 22 is the Hamilton-Jacobi equation of a simple pendulum with its two main families of solutions:

a. $C >\mid \varepsilon A_1 \mid$. In this case the square root never vanishes and keeps a constant sign. $S_1(\theta_1)$ is a monotonically increasing (or decreasing) function. The function $S_1(C, \theta_1)$ is periodic, with period 2π, and this solution is said to be *circulatory*.

b. $\mid C \mid < \mid \varepsilon A_1 \mid$. In this case the square root is not defined for the values of θ_1 such that $C < \varepsilon A_1 \cos\theta_1$. The function S_1 then is not defined for these values of θ_1, which cannot be reached. The function $S_1(C, \theta_1)$ is periodic but with a period smaller than 2π and this solution is said to be *libratory*

In the limiting case $C = \mid \varepsilon A_1 \mid$, the square root vanishes for $\theta_1 = 0$ or $\theta_1 = \pi$ (according to $A_1 > 0$ or $A_1 < 0$, respectively). If, for example, $A_1 > 0$, eqn. 23 becomes

$$\frac{\partial S_1}{\partial \theta_1} = \sqrt{\frac{4\varepsilon A_1}{m_1}} \sin\frac{\theta_1}{2} \tag{24}$$

or

$$S_1 = -\sqrt{\frac{16\varepsilon A_1}{m_1}} \cos\frac{\theta_1}{2}. \tag{25}$$

Again, S_1 is a periodic function of θ_1 but the period is now 4π.

These solutions may be represented geometrically in the polar coordinates x_1, θ_1 (assuming that x_1^* is finite). The case $m_1 > 0, A_1 > 0$ is shown in Figure 1.

3 The role of the square root of the small parameter

The function S_1 of the preceding section is of the order $\mathcal{O}(\sqrt{\varepsilon})$. This fact is a consequence of the inclusion of the quadratic term in the expansion of the functions of x_1 about x_1^* and of the vanishing of n_1. In this sense, eqn. 22 may be seen as a result of the matching of the leading terms in ε and $\frac{\partial S_1}{d\theta_1}$ of the algebraic equation 17, a demarche very usual in the Weierstrass' functions theory (see e.g. Forsyth, 1900, part II). Instead of it, we could obtain this same result as a straightforward application of the implicit functions theorem to equation 17. Indeed, this equation may be written

$$\mathcal{F}(\varepsilon, y) = 0, \tag{26}$$

where, for sake of simplicity, we wrote y for $\frac{\partial S_1}{\partial \theta_1}$.

We have

$$\mathcal{F}(0,0) = 0 \tag{27}$$

(when $\varepsilon = 0$, S degenerates into the Jacobian generator of an identity and S_1 vanishes).

We also have

$$\frac{\partial \mathcal{F}}{\partial \varepsilon}(0,0) \neq 0 \tag{28}$$

and, because of the choice $n_1 = 0$,

$$\frac{\partial \mathcal{F}}{\partial y}(0,0) = 0. \tag{29}$$

However,

$$\frac{\partial^2 \mathcal{F}}{\partial y^2}(0,0) = m_1, \tag{30}$$

which is, generally, finite; the origin is, then, a branch point of the function $y = y(\varepsilon)$ which is developable into a convergent series in the powers of $\sqrt{\varepsilon}$:

$$y = \frac{\partial S_1}{\partial \theta_1} = \sum_k y_k \varepsilon^{k/2} \tag{31}$$

4 The method of Delaunay extended to n degrees of freedom

In this section we consider the solutions of the generalized case

$$F = F_0(x_1) + F_2(x, \theta), \tag{32}$$

in a domain of the phase space including the plan $x_1 = x_1^*$ where $n_1 = 0$. For simplicity, we introduced

$$F_2(x, \theta) = \varepsilon \sum_{k \in \mathbf{D}} A_k \cos(k \mid \theta). \tag{33}$$

It is worth emphasizing that any perturbed system with *one* resonance relation may be reduced to this form by means of a sequence of operations similar to Delaunay's or by using a method of averaging over the high frequencies, as the method of Von Zeipel (see Ferraz-Mello, 1989).

In order to extend to this situation the method discussed in the previous sections, we introduce the Jacobian generating function

$$S(C, \tilde{x}^*, \theta) = (x^* \mid \theta) + S_1 + S_2 + \cdots, \tag{34}$$

where $S_k = \mathcal{O}(\varepsilon^{k/2})$, x_1^* is constant, and

$$\tilde{x}^* \equiv (x_2^*, ..., x_n^*) \tag{35}$$

is the projection of the vector x^* over the hyperplane $x_1^* = 0$. The transformation defined by S is

$$x_i = \frac{\partial S}{\partial \theta_i} = x_i^* + \frac{\partial S_1}{\partial \theta_i} + \frac{\partial S_2}{\partial \theta_i} + \cdots$$

$$\gamma = \frac{\partial S}{\partial C} = \frac{\partial S_1}{\partial C} + \frac{\partial S_2}{\partial C} + \cdots \tag{36}$$
$$\theta^*_\mu = \frac{\partial S}{\partial x^*_\mu} = \theta_\mu + \frac{\partial S_1}{\partial x^*_\mu} + \frac{\partial S_2}{\partial x^*_\mu} + \cdots$$

$(i = 1, 2, ..., n; \mu = 2, 3, ..., n)$ .As this transformation is time- independent, it conserves the Hamiltonian, that is, $F = F^*$, or

$$F_0(x_1) + F_2(x, \theta) = F^*(C, \tilde{x}^* \tilde{\theta}^*), \tag{37}$$

where

$$\tilde{\theta}^* \equiv (\theta^*_2, ..., \theta^*_n) \tag{38}$$

(a similar definition is adopted for $\tilde{\theta}^*$).

The new Hamiltonian F^* is an indeterminate function and is chosen to be independent of γ and to have the form

$$F^* = F^*_0(x^*_1) + C + F^*_2(x^*, \tilde{\theta}^*) + F^*_3(C, x^*, \tilde{\theta}^*) + \cdots \tag{39}$$

C is of order $\mathcal{O}(\varepsilon)$ (as in section 2) and the functions F^*_k are of order $\mathcal{O}(\varepsilon^{k/2})$. We have indicated that F^*_k is also a function of x^*_1 but we remind that this parameter is, now, just a constant, not a canonical variable.

Expanding both sides of eqn. 37 about $x = x^*$ and $\tilde{\theta}^* = \tilde{\theta}$, there follows

$$F_0(x^*_1) + \frac{1}{2} m_1 (\frac{\partial S_1}{\partial \theta_1})^2 + m_1 \frac{\partial S_1}{\partial \theta_1}\frac{\partial S_2}{\partial \theta_1} + \frac{1}{6}\frac{d^3 F_0}{dx^{*3}_1}(\frac{\partial S_1}{\partial \theta_1})^3 + F_2(x^*, \theta) + \sum_i \frac{\partial F_2}{\partial x^*_i}\frac{\partial S_1}{\partial \theta_i} + \mathcal{O}(\varepsilon^2) =$$

$$F^*_0(x^*_1) + C + F^*_2(x^*, \tilde{\theta}) + \sum_\mu \frac{\partial F^*_2}{\partial \theta_\mu}\frac{\partial S_1}{\partial x^*_\mu} + F^*_3(C, x^*, \tilde{\theta}) + \mathcal{O}(\varepsilon^2) \tag{40}$$

(all functions in the partial derivatives in this equation and in the next ones are assumed as functions of (C, x^*, θ)).

Equating the terms of the same order in eqn. 40 we obtain the perturbation equations of the extended Delaunay method:

$$F_0(x^*_1) = F^*_0(x^*_1)$$
$$\frac{1}{2} m_1 (\frac{\partial S_1}{\partial \theta_1})^2 + F_2(x^*, \theta) = C + F^*_2(x^*, \tilde{\theta}) \tag{41}$$
$$m_1 \frac{\partial S_1}{\partial \theta_1}\frac{\partial S_2}{\partial \theta_1} + \frac{1}{6}\frac{d^3 F_0}{dx^{*3}_1}(\frac{\partial S_1}{\partial \theta_1})^3 + \sum_i \frac{\partial F_2}{\partial x^*_i}\frac{\partial S_1}{\partial \theta_i} = \sum_\mu \frac{\partial F^*_2}{\partial \theta_\mu}\frac{\partial S_1}{\partial x^*_\mu} + F^*_3(C, x^*, \tilde{\theta}),$$

etc.

The main equation of the method, the Poincaré-Delaunay equation

$$\frac{1}{2}m_1(\frac{\partial S_1}{\partial \theta_1})^2 + F_2(x^*,\theta) = C + F_2^*(x^*,\tilde{\theta}), \tag{42}$$

is now much more involving than that of section 2. As the aim of the operation under study is only to obtain a new Hamiltonian independent of the angle θ_1, this equation may be split in two parts:

$$\frac{\partial S_1}{\partial \theta_1} = \sqrt{\frac{2}{m_1}\left(C - \sum_{k\in \mathbf{D}_1} \varepsilon A_k(x^*)\cos(k \mid \theta)\right)} \tag{43}$$

and

$$F_2^* = \sum_{k\in \mathbf{D}_2} \varepsilon A_k(x^*)\cos(k \mid \theta). \tag{44}$$

In $\mathbf{D}_1$ we keep all vectors $k \in \mathbf{D}$ whose first component, k_1, is not zero and, in $\mathbf{D}_2$, the rest of them. The solution of eqn. 43 is similar to that discussed in section 2. However, it will depend on the *parameters* x_μ^*, θ_μ. As a consequence, the kinds of motion – classed as circulations and librations in the simple situation of section 2 – will not be invariant, the solutions being able to change their kind as x_μ^*, θ_μ evolve in the time. Equations 41 define the Jacobian generator of a canonical transformation, S, and the new Hamiltonian, $F^*(C,\tilde{x}^*,\tilde{\theta}^*)$, defining a system of equations reduced to $n-1$ degrees of freedom. If we know how to integrate this system, obtaining

$$x_\mu^* = x_\mu^*(t) \qquad \theta_\mu^* = \theta_\mu^*(t), \tag{45}$$

we may introduce these functions into the additional equation

$$\dot{\gamma} = -\frac{\partial F^*}{\partial C} \tag{46}$$

and obtain $\gamma(t)$. We remind that F^* does not depend on γ and that, as a consequence, C is constant. The inverse transformation $(C,\gamma,\tilde{x}^*,\tilde{\theta}^*) \Rightarrow (x,\theta)$ will, then, lead to the solution of the given problem:

$$x = x(t) \qquad \theta = \theta(t) \tag{47}$$

At least schematically, the method of Delaunay, as described here, may yield formal solutions of the given problem.

We have to emphasize that the introduction of the constant C allows x_1^* to be kept fixed. We also stress the fact that it is not necessary to let x_1^* vary to describe the solutions in an interval of x_1 about the fixed value x_1^*. In this sense, the question addressed by Poincaré in the opening of §201 seems to be unnecessary.

5 The singularity of Poincaré

The final analysis of the preceding section supposes that we may solve all equations of the method. In fact, this is not so obvious. Poincaré has shown (*op.cit* p.323) that the equations for $S_q (q \geq 2)$ may be singular. For instance, for $q = 2$, we have

$$\frac{\partial S_2}{\partial \theta_1} = \frac{1}{m_1}(\frac{\partial S_1}{\partial \theta_1})^{-1} \left(F_3^* - \frac{1}{6}\frac{d^3 F_0}{dx_1^{*3}}(\frac{\partial S_1}{\partial \theta_1})^3 + \frac{\partial F_2}{\partial x_1^*}\frac{\partial S_1}{\partial \theta_1} - P_3 \right), \tag{48}$$

where

$$P_3 = \sum_\mu (\frac{\partial F_2}{\partial x_\mu^*}\frac{\partial S_1}{\partial \theta_\mu} - \frac{\partial F_2^*}{\partial \theta_\mu}\frac{\partial S_1}{\partial x_\mu^*}). \tag{49}$$

Eqn. 48 is singular for $\frac{\partial S_1}{\partial \theta_1} = 0$, and this limit is actually reached in all solutions of eqn. 43 classified as librations, i.e., those for which the square root in eqn. 43 is not defined for all θ_1 vanishing for particular values of this angle (libration *boundaries*)

As Poincaré considers only the case with one degree of freedom, he succeeds to get rid of it. Indeed, if we have just one degree of freedom, the variables x_μ, θ_μ do not exist. Then, $P_3 \equiv 0$, and eqn. 48 becomes

$$\frac{\partial S_2}{\partial \theta_1} = \frac{F_3^*}{m_1}(\frac{\partial S_1}{\partial \theta_1})^{-1} - \frac{1}{6m_1}\frac{d^3 F_0}{dx_1^{*3}}(\frac{\partial S_1}{\partial \theta_1})^2 + \frac{1}{m_1}\frac{\partial F_2}{\partial x_1^*}. \tag{50}$$

The singularity disappears if we choose $F_3^* = 0$.

However, in the general case, when the given Hamiltonian has more than one degree of freedom, it is not possible to eliminate the singularity just through an appropriate choice of F_3^*, since this indeterminate function cannot depend on θ_1. In order to remove the singularity we need to change the integration variable, following ideas close to those developed by Hori (1966) in his Lie-series method.

6 Sessin's integration algorithm

The algorithm introduced by Sessin to remove the singularity of Poincaré was founded on the study of the particular one-degree-of- freedom dynamical system whose Hamilton-Jacobi equation is eqn. 42. It is equivalent to solving the partial differential equations of the method of Delaunay by using the method of Cauchy's characteristics

If we introduce the notations

$$y_p = \frac{\partial S_p}{\partial \theta_1}, \tag{51}$$

and write equations 41 generically as $\mathcal{F}(\theta_1, y_p, S_p) = 0$, the equations of their characteristics are (see Carathéodory, 1965):

$$
\begin{aligned}
\frac{d\theta_1}{du} &= \frac{\partial \mathcal{F}}{\partial y_p} \\
\frac{dy_p}{du} &= -\frac{\partial \mathcal{F}}{\partial \theta_1} - y_p \frac{\partial \mathcal{F}}{\partial S_p} \\
\frac{dS_p}{du} &= y_p \frac{\partial \mathcal{F}}{\partial y_p}.
\end{aligned}
\tag{52}
$$

For $p = 1$ (Delaunay-Poincaré equation), the equations of the characteristics are

$$
\begin{aligned}
\frac{d\theta_1}{du} &= m_1 y_1 \\
\frac{dy_1}{du} &= -\frac{\partial F_2}{\partial \theta_1} \\
\frac{dS_1}{du} &= m_1 y_1^2.
\end{aligned}
\tag{53}
$$

From the first two of these equations, we obtain

$$\frac{d^2\theta_1}{du^2} = -m_1 \frac{\partial F_2}{\partial \theta_1}, \tag{54}$$

whose integration is theoretically possible. In particular, it yields the energy-like integral

$$\frac{1}{2}(\frac{d\theta_1}{du})^2 + m_1 F_2 = E, \tag{55}$$

where the integration constant E may be related to the variables by combining this equation and the Delaunay-Poincaré equation:

$$E = m_1(C + F_2^*). \tag{56}$$

The solutions of eqn. 54 are either circulations – when E is greater than the maximum of F_2 – or librations about the minima of that function.

Substituting the value of y_1^2, obtained from eqn. 55, into the third of eqns. 54, we obtain

$$\frac{dS_1}{du} = 2m_1(E - m_1 F_2) \tag{57}$$

or, taking into account eqn. 56,

$$\frac{dS_1}{du} = 2(C + F_2^* - F_2). \tag{58}$$

This equation has still the indeterminate function F_2^* to be fixed before the integration. In the techniques for averaging Hamiltonian systems, like the method of Hori, it is suggested to fix this function in such a way that no secular term exist in S_1. That is, F_2^* may be such that $< \frac{dS_1}{du} >= 0$. This choice is not possible here. Indeed, the function in the right-hand side of the third of eqns. 53 is sign-definite and cannot have a zero average (unless it is identically zero).

Therefore, it is not possible to avoid the existence of secular terms in the generating function. But its geometric propagation may be avoided by choosing it to be independent of x_μ^*, θ_μ. This can be done by adopting $< \frac{dS_1}{du} >= 2C$, that is, $F_2^* =< F_2 >$. Hence, the only derivative of S_1 affected by the secular term is the derivative with respect to C. In the transformations, the only consequence lies in the equation for the new time-like variable γ (see eqns. 36) which will be given by

$$\gamma = \frac{\partial S_1}{\partial C} + ... = u + ... \tag{59}$$

We recall that the derivative of S_1 with respect to C does not appear in the higher-order equations and, as a consequence, the secular term Cu of S_1 does not propagate. We consider, now, the equations for the case $p = 2$. The first of these equations is

$$\frac{d\theta_1}{du} = m_1 y_1,$$

the same as for $p = 1$; the third one is

$$\frac{dS_2}{du} = m_1 y_1 y_2. \tag{60}$$

The second equation, giving the derivative of y_2, does not need to be written, since y_2 is already known from the given eqn. 48:

$$y_2 = \frac{F_3^* - P_3}{m_1 y_1} - \frac{1}{m_1}\left(\frac{1}{6}\frac{d^3 F_0}{dx_1^{*3}} y_1^2 - \frac{\partial F_2}{\partial x_1^*}\right). \tag{61}$$

Hence

$$\frac{dS_2}{du} = F_3^* - \frac{1}{6}\frac{d^3 F_0}{dx_1^{*3}} y_1^3 + \frac{\partial F_2}{\partial x_1^*} y_1 - P_3, \tag{62}$$

whose right-hand side is, now, finite.

Higher-order equations may be treated almost in the same way. After substitution, in turn, of the results obtained in the previous orders, the generic equation may be written

$$m_1 y_1^k y_p = \mathcal{P}(y_1; \theta), \tag{63}$$

where $\mathcal{P}$ is a polynomial in y_1 whose coefficients are functions of x^* and θ; k is an odd integer.

7 Final remarks

The method of Delaunay has been included by Poincaré in his chapter on the method of Bohlin just as an introduction to some of the difficulties found in the study of the perturbations of resonant dynamical systems and is far from complete.

In this lecture we presented the main results found in the book and their extension to situations close to that of real problems. Notwithstanding the fact that this method has not yet been successfully applied to any real problem, it must be considered as an alternative tool to study resonant dynamical systems in quest of good approximations for their phase portrait. It is worth recalling that the Jacobian canonical transformations used in Delaunay method are not homeomorphisms and can provide us with solutions not able to be simply obtained with the use of the techniques founded on Lie series mappings.

One last point to be made is related to the adopted restriction to the case of just one resonance. In fact, almost every step, in the last three sections,may be done by assuming more than one resonance relation,say, $n_\lambda = 0(\lambda = 1, ..., \ell)$. However, it would not be useful. Indeed, in this case, the equations of the characteristics of the Delaunay-Poincaré equation are non-integrable dynamical systems with ℓ degrees of freedom. In the most simple known case, this system is the 2- pendulum, a non-integrable system defined by the Hamiltonian

$$H = F_0(x_1^*, x_2^*) + \varepsilon A_1 \cos\theta_1 + \varepsilon A_2 \cos\theta_2, \tag{64}$$

whose solutions were discussed by Yokoyama(1983) and Lacaz(1985).

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Chapter XIX. Bohlin Method.

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CHAPTER XIX - BOHLIN METHODS

INTRODUCTION

This is by far the longest chapter in Volume 2. In the following personal interpretation of Poincaré's work, I attempt to include all the important aspects of the various methods he desribes. Some of these aspects feature more prominently than others; the choice of these is entirely my own, and I realise others might make a different selection. Where I believe the material is more readily understood by means of some particular example, I include such an example. Certain parts of Poincaré's work have close connections with more recent research of others; where appropriate I make reference to these relationships.

While it is my remit to concentrate my attention on Sections 204-212, I believe it is helpful to give a brief summary of the earlier Sections (§199-§203) of Chapter XIX. I find, for example, that I am unable to reproduce some of Poincaré's formulas of §201, concerning the passage from shallow to deep resonance. (The adjectives shallow and deep are here used in the sense of Garfinkel). Poincaré arrives at similar formulas later in the chapter (§211), where he is considering a more general case of resonance problem. Again, I disagree with his results, but it is easier to outline my disagreement with the aid of the simpler problem of §201.

It is my opinion that §206 contains the most interesting and valuable material of the whole chapter. Apart from my own employment of the techniques described in this section, I am not aware that others have taken advantage of the powerful methods here advocated by Poincaré.

In the following, each section is explained in turn and, where they have been found, minor errors in the original text are indicated at the end of the section. All references are listed at the end of my contribution. Where Poincaré has numbered his equations, e.g. (4), I retain the same label, even though at times his numbering system is either inconsistent or misleading or both. Where I choose to number another equation, whether it be one of my own or one of Poincaré's un-numbered equations, I use the square-bracket notation, e.g. [4].

Delaunay's Method (Comprising §§ 199- 203)

§199 (pp. 315-320)

The chapter begins with a general definition of the resonance problem which is the subject of the whole of Chapter XIX. The problem concerns a canonical system of n degrees of freedom, but with a single critical argument.

Denoting the n-vector $(x_1, x_2, \ldots, x_n)$ by (x) the system may be described by the following Hamiltonian equations:

$$F(x,\chi) = F_o(x) + \mu F_1(x,\chi) + \mu^2 F_2(x,\chi) + \ldots = C = \text{constant};$$

$$\chi = m_1 y_1 + \ldots + m_n y_n \; ; \quad m_i = \text{integer}, \; i = 1(1)n \qquad [1]$$

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i} \, , \quad \frac{dy_i}{dt} = = \frac{\partial F}{\partial x_i} \, .$$

It is assumed that the F_i $(i \geq 1)$ are periodic in χ, with period 2π, and that $|\mu| \ll 1$. The resonance is associated with the critical argument χ, in that the sum

$$n_1^0 m_1 + \ldots n_n^0 m_n$$

is either small or zero. In this combination the frequencies n_i^0 are defined, in the usual way, by

$$n_i^0 = - \left[\frac{\partial F_0}{\partial x_i}\right]_0 ,$$

in which $[\,]_0$ indicates that the substitution $x_i \to x_i^0$ has been made. Here the constants x_i^0 refer to the solution of the unperturbed problem corresponding to $F = F_0$; i.e. the solution corresponding to $\mu = 0$.

The so-called Delaunay method which Poincaré next alludes to is hardly a method as such, but merely an outline of an alternative means of approaching the problem of resonance. Poincaré describes Delaunay's method first as, he says, it pre-dates his own and simplifies the understanding of the latter.

Having formulated his general resonance problem Poincaré chooses first to investigate more closely the simpler one-degree-of-freedom simple pendulum problem,

$$F = x_1^2 + \mu \cos y_1 = C .$$

Introducing the generating function S by $x_1 = \partial S/\partial y_1$, the solution of the resulting Hamilton-Jacobi equation is

$$\frac{\partial S}{\partial y_1} = \sqrt{(C-\mu \cos y_1)}.$$

Poincaré distinguishes three cases, each of which is relevant to later sections in this chapter; they are:

(1) $C > |\mu|$ - the "ordinary" case, which is now more usually called the <u>circulation</u> case,

(2) $-|\mu| < C < |\mu|$ - the <u>libration</u> case,

(3) $C = |\mu|$ - the <u>limiting</u> or <u>separatrix</u> case.

He shows how, in each case, S may be written in periodic form in an appropriate argument. He refers to the use of (i) elliptic functions, which are doubly periodic and (ii) the phase - plane figure of the system - in appropriate "polar" co-ordinates (his fig. 2). Poincaré concludes that while S is expandable in powers of μ, provided $|\mu|/C < 1$, for smaller values of C, i.e. when C is of order μ, then S is of order $\sqrt{\mu}$.

Errors:

p. 317, ℓ. 17 : should read $\frac{\partial S}{\partial y_1} = \sqrt{C(1-\cos\varepsilon)} = \sqrt{2C}\ \sin(\varepsilon/2)$

, ℓ. 19 : ibid $\frac{\partial S}{\partial \varepsilon} = \sqrt{\frac{C(1-\cos\varepsilon)}{\mu^2-C^2\cos^2\varepsilon}}\ C \sin\varepsilon$

, b7 : ibid $\frac{\partial S}{\partial \varepsilon} = B_0 + \Sigma B_n \cos(n\varepsilon/2)$

, b5 : ibid $S = B_0\,\varepsilon + 2\sum(B_n/n)\sin(n\varepsilon/2)$.

p. 318, ℓ2 : should read $\frac{\partial S}{\partial y} = \sqrt{2\mu}\ \sin(y_1/2)$

, ℓ4 : ibid $S = -\sqrt{8\mu}\ \cos(y_1/2)$

p. 319, b11 : should read $C > |\mu|$.

§200 (pp. 320-332)

The subject of this section is the one-degree-of-freedom general problem governed by the Hamiltonian

$$F = F_0(x_1) + \mu F_1(x_1,y_1) + \mu F_2(x_1,y_1) + \ldots\ldots,$$

where the $F_i (i \geq 1)$ are periodic in the argument y_1. Poincaré states that, provided $F'_0(x_1) \neq 0$ then the methods of §125 are applicable; that is, using the appropriate Taylor-series expansion, S may be constructed as a series in powers of μ. He does not state here how small F'_0 is allowed to become before this method is rendered inapplicable.

If $F_0'(x_1) = 0$, however, then it is necessary, in the first instance, to adopt the expansions

$$S = S_0 + \sqrt{\mu}S_1 + \mu S_2 + \ldots \qquad , \quad S_0 = x_1^0 y_1$$

$$C = C_0 + \sqrt{\mu}C_1 + \mu C_2 + \ldots$$

Poincaré proceeds to show how, with a suitable choice of the constants C_p, the generator S and hence the solution can be constructed as series in powers of $\sqrt{\mu}$. He emphasises that with an improper choice of C_p the derivative $\partial S_{p-1}/\partial y_1$, and hence S_{p-1}, may become singular within the libration domain. It is precisely singularities of this kind which Jupp (1970) refers to in his study of the Ideal Resonance Problem. Subsequently, Garfinkel et al. (1971) demonstrate how these "Poincaré" singularities can be avoided; their procedure matches closely Poincaré's prescription. It transpires that $C_{2p-1} = 0$, (for p = 1, 2, ...), S_{2p-1} are series in $\cos(ny_1/2)$ and $\sin(ny_1/2)$, while S_{2p} are series in $\cos ny_1$ and $\sin ny_1$.

§201 (pp. 332-338)

I remarked in the previous section that Poincaré did not state how small F_0' could become before the "classical" method (i.e. that involving expansions in powers of μ) fails. Poincaré takes up this point here, by asking the question: How does one make the passage from the case $F_0' \neq 0$ to the case $F_0' = 0$? . The method he goes on to describe is not attributed specifically to Delaunay, but I infer that this is his intention. The full details of my interpretation of this procedure are published elsewhere (Jupp, 1973), so I limit myself here to a very brief account.

In the "classical" method the generator S is such that

$$S = S_0 + \mu S_1 + \mu^2 S_2 + \ldots \qquad (3)$$

in which S_p contains the factor $(n_1)^{-(2p-1)}$. On writing

$$n_1 = \alpha_1 \sqrt{\mu} + \alpha_2 \mu + \ldots \qquad (2)$$

and substituting into (3), it can be shown that

$$S_p = \sum_{j=1}^{\infty} S_{p,j} \, \mu^{\frac{1}{2}j-p} \quad , \; p \geq 0 \qquad [2]$$

Then placing [2] in (3) yields

$$S = \sum_{p=0}^{\infty} \sum_{j=1}^{\infty} S_{p,j} (\sqrt{\mu})^j = \sum_{p=0}^{\infty} T_p (\sqrt{\mu})^p = T \qquad [3]$$

Thus we have a new expansion of the generator in powers of $\sqrt{\mu}$.

It can readily be shown that

$$T_1 = \frac{-\alpha_1 y_1}{F''_0} + \sum_{p=1}^{\infty} S_{p,1} \, ,$$

where F''_0 is the second derivative of F_0 evaluated at a specific point. Poincaré goes on to state that

$$\left(\frac{\partial T_1}{\partial y_1} \right)^2 = \frac{\alpha_1^{\;2}}{F''^2_0} - \frac{2\alpha_1}{F''_0} \frac{\partial S_{1,1}}{\partial y_1} \, , \qquad [4]$$

i.e. that $(\partial T_1/\partial y_1)^2$ is given by a <u>finite</u> expression ! If this were true then T_1 could, in theory, be expressed in finite form. As is shown in my 1973 publication, I am unable to reproduce this result. Poincaré goes on to state "C'est là un fait d'autant plus remarquable qu'il peut s'éntendre, comme nous le verrons bientôt, à toutes les équations de la Dynamique". Perhaps the error is my own, but so far nobody has come forward to verify this. Poincaré makes no further comment regarding T_2, T_3, ... , as to whether or not they can be written in finite form. As in the case of T_1, I maintain that these quantities are also generally infinite. If my conclusions are correct then the procedure prescribed in this section for making the passage from the classical, non-resonant, case into the domain of resonance is entirely impractical.

The significance of this discrepancy is chiefly academic, for the method of Bohlin, provided it is correctly applied, is a much more practical alternative means of constructing a suitable generating function. Indeed, Garfinkel et al. (1971) show how a "global" solution may be constructed, incorporating both deep and shallow resonance regimes.

§202 (pp. 338-340)

In this very short section Poincaré demonstrates how the results thus far obtained, chiefly relating to one-degree-of-freedom systems, can readily be extended to more general problems. In particular, he mentions the problems associated with

$$\text{(i) } F(x_1, x_2, \ldots, x_n, y_1) = \text{constant}$$

$$\text{(ii) } F(x_1, x_2, \ldots, x_n, \chi) = \text{constant}$$

in which y_1 is the critical argument in the first Hamiltonian and χ, defined by [1], is the critical argument in the second. These extensions are quite trivial; indeed (ii) is simply transformed to (i) by means of a standard linear transformation of variables.

§203 (pp. 341-342)

A summary of the preceding sections is given here, Poincaré considers the general problem associated with

$$F(x_1, x_2, \ldots x_n, y_1, y_2, \ldots, y_n) = \text{constant.}$$

If there are no commensurable relations between the angle variables, then there is no resonance and the methods of §125 are applicable. However, if there is a critical argument, χ say, then the associated term(s) can be 'removed' from the Hamiltonian following the method of Delaunay just described. In other words the critical or resonant part of the Hamiltonian, including all the secular terms, is isolated and integrated separately. Poincaré labels this resonant part F′, and one must bear in mind that in this section F'_0, does not refer to a derivative as it does in earlier and later sections.

<u>Bohlin's Method</u>

§204 (pp. 343-352)

Poincaré says that Delaunay's method, previously described, is inconvenient as it contains, in general, too many changes of variables. I also believe, as I have already stated, that the method is not feasible in most resonance cases.

It is interesting to note that in his introduction to Bohlin's method, Poincaré says that he himself has proposed the same method, but several days later. Is this modesty on Poincaré's part ? Should we, in fact, refer to the technique here described as the Bohlin-Poincaré method ? Indeed, it is not clear at which point in the subsequent sections Poincaré ceases to describe Bohlin's contribution and commences the description of his own. I suspect that most of the material in §205-§212 can be attributed to Poincaré.

The resonance problem considered in this section is characterised by the equations

$$\frac{dx_i}{dt} = \frac{\partial F}{\partial y_i} , \quad \frac{dy_i}{dt} = -\frac{\partial F}{\partial x_i} \tag{1}$$

$$F\left(\frac{\partial S}{\partial y_1} , \ . \ . \ . \ . \ , \frac{\partial S}{\partial y_n} , \ y_1, \ . \ . \ . \ , y_n \right) = C \, . \tag{2}$$

The sum

$$m_1\dot{y}_1 + \ . \ . \ . \ + m_n\dot{y}_n \cong m_1n_1^0 + \ . \ . \ . \ + m_nn_n^0$$

is very small, in which the n_i^0 are the frequencies defined in §199.

Bohlin's method is founded upon the expansions

$$S = S_0 + \sqrt{\mu}\, S_1 + \mu S_2 + \ . \ . \ .$$

$$C = C_0 + C_2\mu + C_4\mu^2 + \ . \ . \ .$$

Such expansions were anticipated in §200. The usual Taylor-series expansion, followed by the separation of terms in equal powers of $\sqrt{\mu}$, generate the equations

$$F_0\left(\frac{\partial S_0}{\partial y_1} , \ . \ . \ . \ , \frac{\partial S_0}{\partial y_n} \right) = C_0$$

$$\left[\sum_i \frac{\partial F_0}{\partial x_i} \frac{\partial S_1}{\partial y_i} \right]_0 = 0$$

$$\left[\sum_i \frac{\partial F_0}{\partial x_i} \frac{\partial S_2}{\partial y_i} + \frac{1}{2} \sum_{i,k} \frac{\partial F_0^2}{\partial x_i \partial x_k} \frac{\partial S_1}{\partial y_i} \frac{\partial S_1}{\partial y_k} + F_1 \right]_0 = C_2 \tag{3}$$

$$\left[\sum_i \frac{\partial F_0}{\partial x_i} \frac{\partial S_3}{\partial y_i} + \frac{1}{2} \sum_{i,k} \frac{\partial^2 F_0}{\partial x_i \partial x_k} \frac{\partial S_1}{\partial y_i} \frac{\partial S_2}{\partial y_k} - \Phi_3 \right]_0 = 0$$

In (3) I have slightly modified Poincaré's notation, in order to clarify the meaning. My brackets $[\,]_0$ denote that the substitutions $x_i \rightarrow x_i^0$ have been made, after the appropriate differentiations have been performed; the constants x_i^0 are defined as in §199. The general equation of (3), that is the equation corresponding to the power $(\sqrt{\mu})^P$, is

$$\left[\sum_i \frac{\partial F_0}{\partial x_i}\frac{\partial S_p}{\partial y_i} + \frac{1}{2}\sum_{i,k}\frac{\partial^2 F_0}{\partial x_i \partial x_k}\frac{\partial S_1}{\partial y_i}\frac{\partial S_{p-1}}{\partial y_k} - \Phi_p\right]_0 = \frac{1}{2}\,(1 + (-1)^P)C_p$$

The notation $\sum_{i,k}$ is to be understood as a sum over both i and k. In all but equation (3c) the combination occurs twice. In equation (3c), the combination appears twice for $i \geq k$, but only once for $i = k$. Where I have written $- F_1$, Φ_3 and Φ_p Poincaré simply has Φ - which might lead to some confusion. In any case, it is assumed always that Φ_p or Φ is a known function.

Let us suppose that the x_i^0 are such that

$$m_1 n_1^0 + . . + m_n n_n^0 = 0; \qquad (4)$$

i.e. there is a close commensurability between the frequencies $\dot{y}_i$. Moreover, we suppose this is the only commensurability of frequencies in the system. The intention is to construct an S such that the $\partial S_p/\partial y_i$ are periodic in the co-ordinates $y_1, y_2, . . . , y_n$. Given that $S_0 = \sum_i x_i^0 y_i$, equation (3a) determines C_0. In virtue of (4), we conclude from (3b) that

$$S_1 = \sum_i \alpha_i y_i + f(\chi), \qquad [5]$$

where the α_i are arbitrary constants, perhaps depending on the x_i^0, and χ is the critical argument defined by [1]. Before proceeding to equation (3c) let us digress a little.

Let U be any function of the n-vector (y), such that all the derivatives $\partial U/\partial y_i$ are periodic functions of the y_i. Thus U is such that it may be expanded in a series whose terms possess one of the following forms:

$\alpha_i y_i$, $\alpha_c \cos(p_1 y_1 + . . + p_n y_n)$, $\alpha_s \sin(p_1 y_1 + . . . + p_n y_n)$.

Denote by $U - \bar{U}$ (Poincaré uses [U] in place of $\bar{U}$) all the trigonometric terms in U <u>except</u> those for which

$$\frac{p_1}{m_1} = \frac{p_2}{m_2} = ... = \frac{p_n}{m_n} .$$

Poincaré calls $\bar{U}$ the mean value of U. In more familiar terminology we would call $\bar{U}$ the long-period or resonant part of U; note that $\bar{U}$ contains the linear (secular) terms in y_i. It is then clearly evident that

$$\overline{\frac{\partial U}{\partial y_i}} = \frac{\partial \bar{U}}{\partial y_i} , \quad \sum_i n_i^0 \frac{\partial \bar{U}}{\partial y_i} = \text{const}, \quad \frac{\overline{U_1 \partial U_2}}{\partial y_i} = \overline{U_1} \, \frac{\overline{\partial U_2}}{\partial y_i} . \qquad [6]$$

In the light of [5],

$$S_1 = \bar{S}_1 ,$$

and so, from [6], the long-period part of the equation (3c) reduces to

$$\left[\text{constant} + \frac{1}{2} \sum_{i,k} \frac{\partial^2 F_0}{\partial x_i \partial x_k} \frac{\partial S_1}{\partial y_i} \frac{\partial S_1}{\partial y_k} + \bar{F}_1 \right]_0 = C_2$$

or

$$\frac{1}{2} \left[\sum_{i,k} \frac{\partial^2 F_0}{\partial x_i \partial x_k} \frac{\partial S_1}{\partial y_i} \frac{\partial S_1}{\partial y_k} \right]_0 = C_2' - [\bar{F}_1]_0 . \qquad (7)$$

Then, with the aid of [5], equation (7) may be written

$$Af'^2 + 2Bf' + D = C_2' - [\bar{F}]_0 , \qquad (8)$$

where the constants A, B and D depend upon the α_i. For $\partial S_1/\partial y_i$ to be periodic in χ it is necessary and sufficient that

$$B^2 - A(D - C_2' + [\bar{F}_1]_0) \geq 0$$

Now the α_i in [5] are arbitrary and, without any loss of generality, we may write

$$\alpha_1 = \alpha_2 = ... = \alpha_n = 0 . \qquad (9)$$

[This statement is verified by Poincaré at the end of this section.]

Incidentally, the choice (9) is tantamount to assuming

$$\frac{\alpha_1}{m_1} = \frac{\alpha_2}{m_2} = \ldots = \frac{\alpha_n}{m_n} \; . \qquad [7]$$

With the choice (9), equation (8) reduces to

$$Af'^2 = C_2' - [\bar{F}_1]_0 \; . \qquad (8.2)$$

Equations [1], [5], (9) and (8.2) lead to the explicit formula for S_1; namely

$$S_1 = \frac{1}{\sqrt{A}} \int \sqrt{C_2' - \left[\bar{F}_1(\chi) \right]_0} \; d\chi \; . \qquad [8]$$

The determination of S_2 is in two parts. First, S_2^* is found from the non-resonant part of (3c); thus we write

$$- \sum_i n_i^0 \left[\frac{\partial S_2^*}{\partial y_i} \right]_0 + \left[F_1^* \right]_0 = 0 \; , \qquad (11)$$

where my asterisk * indicates the non-resonant part; i.e. $U^* = U - \bar{U}$.

Suppose that

$$\left[F_1^* \right]_0 = \sum_p A_p \cos\,(p_1y_1 + p_2y_2 + \ldots\; p_ny_n + \beta_p),$$

where the subscripts p are my own, to indicate that the constants A_p and β_p usually depend upon the combinations of the p_i. It then follows from (11) that

$$S = \bar{S}_2 + S_2^* = \bar{S}_2(\chi) + \sum_p \frac{A_p \sin\left(p_1y_1 + \ldots + p_ny_n + \beta_p\right)}{p_1n_1^0 + \ldots + p_nn_n^0} \; .$$

None of the divisors in this series is zero, because F_1^* contains no term depending on the critical argument χ. Second, to determine $\bar{S}_2$ we must turn to equation (3d) and take its long-period part. Making use of [6] and the fact that $S_1^* = 0$, we deduce that

$$\frac{1}{2}\left[\sum_{i,k}\frac{\partial^2F_0}{\partial x_i\partial x_k}\frac{\partial S_1}{\partial y_i}\frac{\partial\bar{S}_2}{\partial y_k}\right]_0 = \left[\Phi_3\right]_0 . \qquad [9]$$

Hence, since $\partial S_1/\partial y_i = m_i f'$ and $\partial\bar{S}_2/\partial y_k = m_k\bar{S}_2'$, integration of [9] leads to

$$\bar{S}_2 = \alpha_1^{(2)}y_1 + \ldots + \alpha_n^{(2)}y_n + \frac{2}{\sum_{i,k} m_i m_k\left[\partial^2F_0/\partial x_i\partial x_k\right]_0}\int\frac{[\bar{\Phi}_3]_0}{f'}\,d\chi . \qquad [10]$$

In the light of relations [7], let us further assume that the arbitrary constants $\alpha_i^{(2)}$ are chosen such that

$$\frac{\alpha_1^{(2)}}{m_1} = \frac{\alpha_2^{(2)}}{m_2} = \ldots = \frac{\alpha_n^{(2)}}{m_n} . \qquad [11]$$

The consequence of this choice is that

$$-\overline{\sum_i n_i^0\frac{\partial S_2}{\partial y_i}} = -\sum_i n_i^0\frac{\partial\bar{S}_2}{\partial y_i} = 0 , \qquad [12]$$

which, in turn, implies

$$C_2 = C_2'.$$

Thus, in equation [8] for S_1, C_2' may be replaced by C_2.

Now that S_1 and S_2 have been completely determined, S_3^* can similarly be found from the next equation (3e) and $\bar{S}_3$ from equation (3f). The method may be continued to any desired order. We observe that, in general, S_p is determined from two consecutive equations in (3); namely those of orders $\mu^{p/2}$ and $\mu^{(p+1)/2}$. It is easily demonstrated that, at order p, provided

$$\frac{\alpha_1^{(p)}}{m_1} = \frac{\alpha_2^{(p)}}{m_2} = \ldots = \frac{\alpha_n^{(p)}}{m_n} , \qquad (10)$$

in parallel with [11], then

$$- \sum_i n_i^0 \frac{\partial \bar{S}_p}{\partial y_i} = 0 , \qquad [13]$$

which is the generalisation of [12].

Libration Case (Comprising §205, §206)

§205 (pp. 352-354)

It can happen that $C_2 \le \max [\bar{F}_1]_0$. In such cases f′, and therefore $\partial S_1/\partial y_i$, vanishes for some values of y_i. Such a state is known as libration. To investigate this case further let us simplify the system, as we did before in §199; thus, we assume that

$$m_1 = 1, \; m_2 = \ldots = m_n = 0 .$$

Then $\chi = y_1$, so that y_1 is now the critical argument. With reference to the previous section U^* is now a periodic function in y_2, ..., y_n, but not y_1. We found before that, provided equations (9) are chosen to apply, S_1 depends only on χ. In this case the equation equivalent to (7), for the determination of S_1, is

$$\frac{1}{2} \left[\frac{\partial^2 F_0}{\partial x_1^2} \left(\frac{\partial S_1}{\partial y_1} \right)^2 \right]_0 = C_2 - [\bar{F}_1]_0 . \qquad (13)$$

For the calculation of S_2^* equation (11) remains valid, while equation [9] for $\bar{S}_2$ reduces to

$$\left[\frac{\partial^2 F_0}{\partial x_1^2} \right]_0 \frac{\partial \bar{S}_2}{\partial y_1} \frac{\partial S_1}{\partial y_1} = \left[\bar{\Phi}_3 \right]_0 . \qquad (15)$$

[Note that each combination i, k occurs twice in [9], and so the factor 1/2 in [9] is cancelled]. Carrying the procedure to the next orders the functions S_3^*, $\bar{S}_3$, S_4^*, $\bar{S}_4$, . . .are computed in turn.

Error:
p. 353, b7; should read $y_2, y_3, \ldots, y_n$.

§§206 (pp. 354-366)

While Poincaré makes no explicit statement of the fact, it is clear that the equations of the previous section cannot be applied directly in cases of libration, since $\partial S_1/\partial y_1$ vanishes for some values of y_1. Unless appropriate measures are taken the subsequent expressions for the S_p are singular for these same values of y_1. I discovered, quite independently, precisely this circumstance in Garfinkel's original solution of the Ideal Resonance Problem (Garfinkel, 1966). In this solution Garfinkel uses a procedure which he calls the Bohlin-von Zeipel method. The difficulty was subsequently overcome (Garfinkel et al, 1971) by making a "proper"choice of new Hamiltonian and introducing a "regularising" function.

Nonetheless, this difficulty associated with these singularities and another difficulty, referred to as the "loss of an order of magnitude on differentiation with respect to the momentum", renders Garfinkel's revised solution extremely complicated. I maintain that my own solution of the Ideal Resonance Problem, based on the methods Poincaré describes in this section, is significantly simpler and more practical. I briefly outline my method at the end of this section.

In my opinion this section is the most interesting and valuable of the whole chapter. On the other hand, in some respects it is also the most bewildering.

Poincaré begins "Pour étudier plus complètement nos functions, it faut faire un changement de variables". I assume that the functions, referred to are the derivatives $\partial S_p/\partial y_i$. He gives no reason for adopting this particular method of proceeding. As in §205, it is assumed that $m_1 = 1$, $m_2 = \ldots = m_n = 0$. He begins by introducing a finite auxiliary function T, such that

$$T = T_0 + \sqrt{\mu} T_1 + \mu T_2 \qquad [14]$$

with

$$T_0 = S_0 = x_1^0 y_1 + \ldots x_n^0 y_n \qquad [15]$$

He defines T_1 such that

$$\frac{1}{2} \sum_{i,k} \left[\frac{\partial^2 F_0}{\partial x_i \, \partial x_k} \right]_0 \frac{\partial T_1}{\partial y_i} \, \frac{\partial T_1}{\partial y_k} = C_2 - \left[\bar{F}_1 \right]_0, \qquad (7.2)$$

which is equation (7) with S_1 and C_2' replaced by T_1 and C_2 respectively. In addition, he imposes the relation

$$\frac{\partial T_1}{\partial y_i} = x_i' \quad , \quad i = 2(1)n.$$

where the x_i' are constants. It is perhaps noteworthy that in (7) Poincaré writes $\partial^2 F_0/\partial x_i \partial x_k$, while in (7.2) he writes $\partial^2 F_0/\partial x_i^0 \partial x_k^0$. I believe, in both cases, he must mean $\partial^2 F_0/\partial x_i \partial x_k$ evaluated at $x_i = x_i^0$ $(i = 1(1)n)$. It follows that T_1 is of the form

$$T_1 = x_2' y_2 + \ldots + x_n' y_n + \hat{T}_1(y_1).$$

Recall that in the case of S_1, we set the constants α_i equal to zero (equation (9)); here the x_i' are non-zero. Equation (7.2) may now be written

$$A \left(\frac{\partial T_1}{\partial y_1} \right)^2 + 2B \frac{\partial T_1}{\partial y_1} + D = C - [\bar{F}_1]_0 \, , \qquad (8.3)$$

where A is a constant depending on the x_i^0, while B and D depend also on the constants x_i'. Moreover, B is linear and D is quadratic in the x_i'. It follows that

$$\frac{\partial T_1}{\partial y_1} = \frac{-B}{A} + \sqrt{\frac{B^2}{A^2} - \frac{D}{A} + \frac{C_2}{A} - \frac{[\bar{F}_1]_0}{A}} \; .$$

Poincaré next defines x_1' by the formula

$$x_1' = (B^2 - AD + AC_2)/A^2. \qquad [16]$$

There results the formula for T_1; thus,

$$T_1 = x_2' \, y_2 + \; . \; . \; . \; + x_n' \, y_n - \frac{By_1}{A} + \int \sqrt{x_1' - \psi} \; dy_1 \qquad [17]$$

in which ψ is defined by

$$[\bar{F}_1]_0 = A\psi.$$

It is still not at all obvious to me why Poincaré has chosen T_1 in this particular way. Later on it will be seen just how significant the choice is.

Next, T_2 is chosen, in analogy with equation (11), such that

$$\sum_i n_i^0 \frac{\partial T_2}{\partial y_i} = [F_1^*]_0.$$

However, the further constraint

$$\bar{T}_2 = 0 \qquad [18]$$

is imposed. Accordingly, T_2 is independent of the x_i'. Lastly the constant C_1 is defined by

$$C_1 = \sum_i n_i^0 \, x_i' \; ,$$

in which we must remember that $n_1^0 = 0$.

The outcome of all these definitions is that on replacing S by T in F there results

$$F\left[\frac{\partial T}{\partial y_1}, \frac{\partial T}{\partial y_2}, \; . \; . \; . \; , \frac{\partial T}{\partial y_n}, y_1, y_2, \; . \; . \; . \; y_n\right] = C_0 + \sqrt{\mu}\, C_1 + \mu \, C_2 + \left. + \, O(\mu \sqrt{\mu})\right] .$$

The function T, depending as it does on (x') and (y), is used as a generating function by Poincaré to effect the canonical change of variables $(x, y) \longrightarrow (x', y')$ defined by

$$x_i = \frac{\partial T}{\partial y_i}, \quad y_i' = \frac{\partial T}{\partial x_i'}. \qquad [19]$$

In explicit form, from equations [14], [15], [17] and [19], we have

$$x_1 = x_1^0 + \sqrt{\mu}\left(-\frac{B}{A} + \sqrt{x_1' - \psi}\right),$$

$$y_1' = \frac{\sqrt{\mu}}{2}\int \frac{dy_1}{\sqrt{x_1' - \psi}},$$

$$x_i = x_i^0 + x_i'\sqrt{\mu} + \mu\frac{\partial T_2}{\partial y_i}, \qquad i = 2(1)n \qquad (16)$$

$$y_i' = y_i\sqrt{\mu} - \frac{y_1}{A}\sqrt{\mu}\frac{\partial B}{\partial x_i'}, \qquad i = 2(1)n.$$

I believe that Poincaré has here made another error. In equation (16a) he has the additional term $\mu\ \partial T_2/\partial y_1$ in the right-hand side. Yet with the choice [18], T_2 must be independent of the critical argument y_1. Fortunately, the apparent error does not seem to have any significant effect. Poincaré proceeds to draw an analogy between (16b) and the elliptic functions he introduced in §199. He argues that $\sqrt{x_1' - \chi}$ is a periodic function of y_1, with the period proportional to $\sqrt{\mu}$. Further, he remarks that x_1' plays a role analagous to the modulus of the elliptic functions. He concludes that on changing variables from $(x, y) \longrightarrow (x', y')$ the Hamiltonian F becomes a periodic function of the y_i'. The period of these functions is $2\pi\sqrt{\mu}$ for y_i, $i = 2(1)n$, but

$$\sqrt{\mu}\int_{\alpha}^{\beta} \frac{dy_1}{\sqrt{x_1' - \psi}} = \sqrt{\mu}P \qquad [20]$$

for y_1'. The limits of integration are the bounds on y_1 in libration.

The simple scaling change of variables

$$y_i' = \sqrt{\mu} z_i, \qquad [21]$$

yields an F which has period P in z_1 but 2π in z_i, for $i = 2(1)n$.

When F is expressed in terms of x_i' and z_i, the first three terms of the expansion are

$$C_0 + C_1\sqrt{\mu} + C_2\mu,$$

each of which is independent of the z_i. C_0 is an absolute constant, depending on the x_i^0, while C_1 is linear in x_i', $i = 2(1)n$ and, in virtue of [16], C_2 is linear in x_1' but quadratic in the other x_i'. In order to retain the standard form of Hamilton's equations in the transformed variables Poincaré defines H such that

$$F = C_0 + \sqrt{\mu}H . \qquad [22]$$

(Poincaré uses the symbol F^* where I have written H, but I have already used the asterisk for another purpose.) Then

$$\frac{dx_i'}{dt} = \frac{\partial H}{\partial z_i}, \quad \frac{dz_i}{dt} = -\frac{\partial H}{\partial x_i'} . \qquad (17)$$

However, while H has period 2π in the variable z_i, $i = 2(1)n$, it has period P in z_1. To overcome this problem, which would otherwise lead to mixed-secular terms in the later necessary inversion procedures, a simple change of variables is sufficient; thus Poincaré defines the canonical transformation

$(x_1', z_1) \longrightarrow (u_1, v_1)$ by

$$u_1 = \frac{1}{2\pi}\int P dx_1', \qquad v_1 = \frac{2\pi z_1}{P} . \qquad (18)$$

The final set of canonical variables is $(u_1, x_2', \ldots, x_n', v_1, z_2, \ldots, z_n)$. Poincaré asserts that the new Hamiltonian H, when expressed in terms of this set of variables, may be handled by the methods discussed in §134. It is worth noting here that this new Hamiltonian is essentially non-resonant. For while $\partial F_0/\partial x_1 = 0$ and $\partial^2 F_0/\partial x_1^2 \neq 0$ it is easy to show that, if we write

$$H = H_0 + \sqrt{\mu}H_1 + \mu H_2 + \ . \ . \ . \ ,$$

then because $H_0 = C_1$, $\partial^j H_0/\partial u_1{}^j = 0$ for $j = 1, 2 \ldots$.

Further

$$\frac{\partial H_1}{\partial u_1} = \frac{\partial C_2}{\partial x_1'}\frac{dx_1'}{du_1} = \frac{2\pi A}{P} \neq 0 \ .$$

At this juncture I believe it will be instructive to demonstrate an application of the procedures advocated by Poincaré in this section. The Ideal Resonance Problem (I.R.P.) first formulated by Garfinkel in 1966, is a special case of the problem under consideration in this section. In my original solution (Jupp, 1969) of this problem I made use, to great advantage, of Poincaré's techniques. I will therefore briefly explain my approach. In order not to confuse the reader with symbols already used in my description of Poincaré's works I elect to modify the symbols I used in 1969.

The I.R.P., in its simplest form, is a one-degree-of-freedom system governed by the equations

$$F = b(x) + 2\mu\, a(x) \sin^2 y, \qquad \mu << 1$$

$$\frac{dx}{dt} = \frac{\partial F}{\partial y}, \quad \frac{dy}{dt} = -\frac{\partial F}{\partial x} \ . \qquad [23]$$

(In the original publication I used A(x) and B(x) rather than a(x) and b(x)). Resonance is associated with the vanishing of $\partial b/\partial x$ for some value of x; x_0 say. The direct Bohlin method applied to this problem runs into difficulties in cases of libration, because of the appearance of the Poincaré-type singularities mentioned in §205. (There are also other significant difficulties with this direct method.)

Following the ideas of Poincaré I make three transformations of variables as described below.

Transformation 1: $(x,\ y) \rightarrow (p,\ q)$.

$$p = \left(\frac{x - x_0}{\mu} \right)^2 + p_0 \sin^2 y$$

$$q = \left(\frac{1}{2} \sqrt{\mu} \right) \int [p - p_0 \sin^2 y]^{-\frac{1}{2}} dy. \qquad [24]$$

in which p_0 is a constant yet to be defined.

By making the identifications of p, q, x_0 and $p_0 \sin^2 y$ with x_1', y_1', x_1^0 and ψ respectively, it is readily seen that [24a, b] are equivalent to (16a, b). Moreover, in this application $B \equiv 0$. Then, writing

$$F\ (x,\ y) = \Phi\ (p,\ q), \qquad [25]$$

and taking into account that $\partial b/\partial x = 0$ at $x = x_0$, a Taylor-series expansion yields

$$\Phi(p,q) = b_0 + \sum_{n-2}^{\infty} (\mu p)^{\frac{n}{2}} \left[\frac{b_0^{(n)}}{n!} cn^2 w + \frac{2a_0^{(n-2)}}{p_0 (n-2)!} sn^2 w \right] cn^{n-2} w, \qquad [26]$$

There is no term in $\sqrt{\mu}$ since $b_0^{(1)} = n_1^{\ 0} = 0$. The notation $b_0^{(n)}$ means $\partial^n b/\partial x^n$ evaluated at $x = x_0$, and sn and cn are the Jacobi elliptic functions with modulus k and argument w given by

$$k = \sqrt{p/p_0}, \qquad w = 2 \left(\frac{p_0}{\mu} \right)^{\frac{1}{2}} q. \qquad [27]$$

The constant p_0 is now chosen so that the leading term in the infinite series in [26] is independent of w(i.e. q). Accordingly, we choose

$$p_0 = 4a_0/b_0^{(2)}. \qquad [28]$$

Since the I.R.P. described here has just one degree of freedom, equations (16c,d) are not relevant.

Transformation 2: $(p(t),\ q(t)) \rightarrow (p(\tau),\ s(\tau))$:

The relations
$$s = q\mu^{-1/2}, \qquad [29]$$
$$\tau = t\mu^{1/2} \qquad [30]$$
$$\Phi = b_0 + \mu\Psi \qquad [31]$$

ensure that the transformed equations of motion are in the standard form

$$\frac{dp}{d\tau} = \frac{\partial\Psi}{\partial s},\quad \frac{ds}{d\tau} = -\frac{\partial\Psi}{\partial p}.$$

We observe that [29] corresponds to equation [21], while [31] is similar to [22], but with μ as a factor rather than $\sqrt{\mu}$. This is because in this application $C_1 = 0$. In order to retain the standard form of Hamilton's equations the time has been scaled by the factor $\sqrt{\mu}$.

Transformation 3: $(p,\ s) \rightarrow (u,\ v)$

Returning to equation [20], in which the period P of libration is defined, and in view of [27a], for the I.R.P. we obtain

$$P = \int_{-\sin^{-1}k}^{\sin^{-1}k} \frac{dy}{\sqrt{p-p_0}\ \sin^2 y}$$

Simple integration yields

$$P = 4K/p_0^{1/2}$$

where K is the complete elliptic integral of the first kind.

Accordingly, the transformation equations (18) become, in the case of the I.R.P.,

$$u = \frac{2}{\pi p_0^{1/2}} \int K \, dp, \quad v = \frac{\pi p_0^{1/2} s}{2K} . \qquad [32]$$

Writing $p = p_0 k^2$, [32a] may be written

$$u = \frac{4p_0^{1/2}}{\pi} \int Kkdk = \frac{4p_0^{1/2}}{K} (E - k'^2 K) \qquad [33]$$

.

in which E is the complete elliptic integral of the second kind and k' is the complementary modulus. Equation [33] relates the 'new' momentum variable u to the elliptic modulus k.

Summary of the I.R.P. transformation.

The 3 transformations just described may be condensed to the single canonical transformation $(x, y) \longrightarrow (u, v)$ defined by

$$x = x_0 + (\mu p_0)^{1/2} k\,\mathrm{cn}(w, k),$$

$$\sin y = k\,\mathrm{sn}(w, k), \qquad [34]$$

with $$w = 4Kv/\pi.$$

The modulus k and complete elliptic integral K depend on u, according to equation [33]. The transformation [34], in view of [26] and [31], furnishes the new Hamiltonian.

$$\Psi = \frac{p_0 b_0^{(2)} k^2}{2} + p_0^{1/2} \sum_{n=3}^{\infty} (\mu p_0)^{n/2} k^n \left[\frac{b_0^{(n)}}{n!} \mathrm{cn}^2 w + \frac{2a_0^{(n-2)}}{p_0 (n-2)!} \mathrm{sn}^2 w \right] \mathrm{cn}^{n-2} w$$

$$= \Psi_0 + \sqrt{\mu}\, \Psi_1 + \mu\, \Psi_2 + \ldots \qquad [35]$$

The leading term Ψ_0 is such that

$$\frac{\partial \Psi_0}{\partial u} = \frac{d\Psi_0}{dk} \frac{dk}{du} = \sqrt{a_0 b_0^{(2)}} \; \frac{\pi}{2K} . \qquad [36]$$

In libration $0 < k < 1$ and so $\partial\Psi_0/\partial u$ is non-zero. Thus Ψ is essentially a non-resonant Hamiltonian. The transformed system of equations

$$\frac{du}{d\tau} = \frac{\partial\Psi}{\partial v}, \qquad \frac{dv}{d\tau} = \frac{-\partial\Psi}{\partial u} \qquad [37]$$

may be formally solved using either the standard von Zeipel procedure (Jupp, 1969) or a method based on Lie series (Jupp, 1973). In either case the solutions are constructed as series in powers of $\sqrt{\mu}$. It is important to understand, however, that this last procedure is not what is usually referred to as Bohlin's method. For in arriving at [37] with Ψ as defined in [35] we have transformed the problem into one of the type which Poincaré discusses in §125 - but with $\sqrt{\mu}$ replacing the μ of §125.

Let us return to Poincaré's text on page 360. I find the remainder of this section quite difficult; Poincaré's line of reasoning is not made clear to simple-minded people like myself. Nevertheless, I will endeavour to explain the text as best I can; but I sometimes choose to deviate from his order of presentation.

Poincaré's successive changes of variables just described have transformed the problem into one involving the new set of variables $(u, x_2', \ldots x_n', v_1, z_2, \ldots, z_n)$. Let the function V be such that

$$H\left(\frac{\partial V}{\partial v_1}, \frac{\partial V}{\partial z_2}, \ldots, \frac{\partial V}{\partial z_n}, v_1, z_2 \ldots z_n\right) = \text{const.} \qquad (23)$$

with

$$u_1 = \frac{\partial V}{\partial v_1}, \quad x_i' = \frac{\partial V}{\partial z_i} \;; \quad i = 2(1)n.$$

In other words let V take on the role with H as S did with F (cf. equation (2) of §204). Then V can be determined in the form.

$$V = V_0 + \sqrt{\mu}V_1 + \mu V_2 + \ldots$$

with

$$V_i = \beta_i^1 v_1 + \beta^2_i z_2 + \ldots + \beta^n_i z_n + V_i'. \;, \quad i = 1(1)n.$$

Here the β_i^p are constants and the V_i' are periodic, with period 2π, in v_1 and z_i, with the exception that $V_0' \equiv 0$. It follows that

$$dV = \frac{\partial V}{\partial v_1} dv_1 + \sum_{i=2}^{n} \frac{\partial V}{\partial z_i} dz_i = u_1 dv_1 + \sum_{i=2}^{n} v_i dz_i$$

is an exact differential. The complete transformation $(x, y) \rightarrow (u_1, x', v_1, z)$ is canonical and so the difference

$$\sum_{i=1}^{n} x_i\, dy_i - \sqrt{\mu} \left(u_1\, dv_1 - \sum_{i=2}^{n} v_i\, dz_i \right) = dS - \sqrt{\mu} dV$$

is also an exact differential. The factor $\sqrt{\mu}$ appears because of the second, scaling transformation. We conclude that the S defined here, i.e.

$$dS = \sum_{i=1}^{n} x_i dy_i \qquad (20)$$

is precisely the same as the S of §205. Poincaré thus establishes that the two systems of equations, the one in the original variables, and the other in the transformed new variables, are essentially identical. In the previous sections the constants α_i and α_i^p were chosen according to hypotheses (9) and (10), which did not detract in any way from the generalisation of the procedure. Similarly the β_i^p must satisfy corresponding conditions; they are simply that the x_i' $(i = 2(1)n)$ vanish with $\mu = 0$.

Through the various transformation equations it is possible, in theory, to write

$$y_1 = \theta\ (v_1, y_2, \ldots, y_n)$$
$$x_i = \zeta_i\ (v_1, y_2, \ldots, y_n)$$

where θ and ζ_i are periodic, with period 2π, with respect to each of the n variables. Consider the case corresponding to $y_2, \ldots, y_n$ being constants; then the equations

$$y_1 = \theta(v_1), \; x_1 = \zeta_1(v_1)$$

describe a closed curve in the (x_1, y_1) plane as v_1 increases by 2π. Eliminating v_1 is equivalent to writing the exact equation

$$x_1 = \frac{\partial S}{\partial y_1},$$

which again describes a closed curve. [This must be so since we are throughout this section considering libration.] On the other hand, following the procedure described in §205 we construct x_1 as follows:

$$x_1 = \frac{\partial S_0}{\partial y_1} + \sqrt{\mu}\,\frac{\partial S_1}{\partial y_1} + \mu\,\frac{\partial S_2}{\partial y_1} + \ldots + \mu^{p/2}\,\frac{\partial S_p}{\partial y_1}. \qquad (26)$$

In theory the series is infinite, and divergent, but in practice we must choose a term at which to terminate the series. Poincaré asks whether the curve, in the (x_1, y_1) plane, given by (26), is closed.

As I remarked at the beginning of this section, it is not generally possible, at all orders, to construct $\partial S_p/\partial y_1$, which remains finite within the libration domain. Poincaré does not mention this fact before, but he here gives the reason for the singular derivatives. The reason is that, at order p, the resonant part of S_p is calculated from an equation of the form

$$\left[\frac{\partial^2 F_0}{\partial x_1^2}\right]_0 \frac{\partial \bar{S}_p}{\partial y_1}\,\frac{\partial S_1}{\partial y_1} = [\bar{\Phi}_{p+1}]_0 + C_{p+1} \qquad (27)$$

where $C_p = 0$ for p odd; equation (15) is of this form, with $p = 2$. Poincaré asserts that $\partial S_2/\partial y_1$ remains finite because $[\bar{\Phi}_3]_0$ vanishes with $\partial S_1/\partial y_1$; this is indeed true, as I have myself verified that, for example, in the one-degree-of-freedom case

$$[\bar{\Phi}_3]_0 = -\left[\frac{\partial \bar{F}_1}{\partial x_1}\right]_0 \frac{\partial S_1}{\partial y_1} - \frac{1}{6}\left[\frac{\partial^3 F_0}{\partial x_1^3}\right]_0 \left(\frac{\partial S_1}{\partial y_1}\right)^3. \qquad [38]$$

However, this is not the case when p = 3. Moreover, it is not possible to choose C_4 to keep $\partial\bar{S}_3/\partial y_1$ finite at both zeros of $\partial S_1/\partial y_1$. Further difficulties of the same kind exist for all odd values of p.

Poincaré chooses to illustrate this situation by means of a simple example. It is obvious that the equation

$$x = \sqrt{1 - y^2 + \mu y^2} \quad \left(= \frac{\partial S}{\partial y} \right)$$

is a closed ellipse. However, the infinite expansion

$$x = (1-y^2)^{1/2} + \frac{\mu}{2}\frac{y^2}{(1-y^2)^{1/2}} - \frac{\mu^2}{8}\frac{y^4}{(1-y^2)^{3/2}} + \ldots$$

does not describe a closed curve, since $x \to \infty$ as $y \to \pm 1$.

Poincaré concludes this section on libration by stating that all such difficulties can be avoided by changing the variables $(x, y) \to (u_1, x', v_1, z)$ as described. I have already shown that, in the case of the I.R.P., his prescription is wholly satisfactory.

<u>Errors</u>:

p. 356, b 6: should read " ... la deuxième équation (16) ...".

p. 357, l 3: should read " ... notre deuxième équation (16) ...".

p. 359, b 13: should read " ... C_2 est un polynôme de premier ordre par rapport à x_1' mais de deuxième ordre par rapport aux autres x_i'.

p. 366, l 2: should read " ... $-\frac{\mu^2}{8}\frac{y^4}{(1-y^2)^{3/2}}$ ".

Limiting (Separatrix) Case (comprising §207-§210)

§207 (pp. 366-368)

The separatrix case corresponds to $C_2 = \max [\bar{F}_1]_0$ in equation (15). We are still assuming $m_1 = 1$, $m_i = 0$ $(i > 1)$ and so $n_1^0 = 0$ and χ $(=y_1)$ is the critical argument. In this limiting case it was shown in §200 that $\partial S_1/\partial y_1$ is periodic in y_1 but with period 4π, rather than 2π. Moreover, there is a single value of y_1 for which $C_2 - [\bar{F}_1]_0$ vanishes. Without loss of generality we choose this value to correspond to $y_1 = 0$. Then,

$$\frac{\partial S_1}{\partial y_1} = 0 \quad \text{for } y_1 = 2k\pi,$$

for any integer k. For example, with $F_1 = \cos y$, and $C_2 = 1$,

$$\frac{\partial S_1}{\partial y_1} = \sqrt{1-\cos y} = \sqrt{2}\,\sin(y_1/2)$$
$$= 0 \text{ for } y_1 = 2k\pi,$$

Next, in view of equations (15) and (27), Poincaré asks whether the $\partial S_p/\partial y_1$ remain finite, given that $\partial S_1/\partial y_1$ vanishes at $y = 2k\pi$. After a lengthy, in depth, analysis he concludes that these derivatives, and hence the S_p do remain finite in the limiting case. But this conclusion is not reached until the end of §210, and for reasons of economy of space I prefer to present here only a brief summary of Poincaré's analysis.

Bohlin's method of §204, prescribes $C_i = 0$ for odd values of i. In equation (27), when p is odd, we may always choose C_{p+1} such that the right-hand side vanishes for $y_1 = 2k\pi$. Accordingly the $\partial S_p/\partial y_1$ remain finite, and so therefore do the $\bar{S}_p$ and S_p. However, this choice is not available to us when p is even in equation (27), for then $C_{p+1} = 0$. To establish that $\partial S_p/\partial y_1$ remains finite when p is even it is necessary to prove that, at $y_1 = 0$,

$$[\bar{\Phi}_{p+1}]_0 = 0 \tag{29}$$

is a self-satisfied relation. Poincaré constructs such a proof in the next three sections.

§208-§210 (pp. 368-383)

Poincarè proves that (29) is self-satisfied in a style which typifies his rigour and enthusiasm for his subject. In his proof he employs material from earlier sections; namely §42, 43, 44, 79, 145 and also results from Chapters VII and XV.

Rather than attempt to present Poincaré's analysis I will simply verify that $\partial S_4/\partial y_1$ remains finite. The case corresponding to p = 2 has already been verified, in that equation [38] illustrates that $[\Phi_3]_0$ is factored by $\partial S_1/\partial y_1$.

Let us consider, without loss of generality, the one-degree-of-freedom case corresponding to Hamiltonian F (x, y), in which F is of the standard form and y is the critical argument. Then $[\partial F_0/\partial x]_0 = F_0' = 0$. The formal Bohlin method provides the set of equations

$$F_0 = C_0,$$

$$\frac{1}{2} F_0'' S_{1y}{}^2 + F_1 = C_2 ,$$

$$F_0'' S_{1y} S_{2y} + \frac{1}{6} F_0''' S_{1y}{}^3 + F_1' S_{1y} = 0, \qquad [39]$$

$$F_0'' S_{1y} S_{3y} + \frac{1}{2} F_0'' S_{2y}^2 + \frac{1}{2} F_0''' S_{1y}{}^2 S_{2y} + \frac{1}{24} F_0'''' S_{1y}^4 + $$
$$+ F_1' S_{2y} + \frac{1}{2} F_1'' S_{1y} + F_2 = C_4,$$

$$F_0'' S_{1y} S_{4y} + F_0'' S_{2y} S_{3y} + \frac{1}{2} F_0''' (S_{1y}^2 S_{3y} + S_{1y} S_{2y}^2) + \frac{1}{6} F_0'''' S_{1y}^3 S_{2y} +$$

$$+ \frac{1}{5!} F_0''''' S_{1y}^5 + F_1' S_{3y} + F_1'' S_{1y} S_{2y} + \frac{1}{2} F_1''' S_{1y}^3 + F_2' S_{1y} = 0,$$

. . . .

In these equations I have omitted, for the purpose of simplification, the brackets $[\,]_0$; it is to be understood that all the functions F_i and their derivatives are evaluated at $x = x^0$, the root of $\partial F_0/\partial x = 0$. Further, as there exists a single degree of freedom we here have $\bar{S}_p = S_p$, since $S_p^* \equiv 0$. The derivatives $\partial S_p/\partial y$ are here abbreviated to S_{py}.

With reference to equation (27), $C_5 = 0$ and the only terms in $[\bar{\Phi}_5]_0$ not explicitly factored by S_{1y} in [39e] I label M; thus

$$M = F_0'' \, S_{2y} \, S_{3y} + F_1' \, S_{3y} \,.$$

However, in virtue of [39c], we see that

$$F_0'' \, S_{2y} = -\frac{1}{6} F_0''' \, S_{1y}^2 - F_1' \,.$$

Accordingly

$$M = -\frac{1}{6} F_0''' \, S_{1y}^2 \, S_{3y} \,.$$

We have demonstrated that every term appearing in [39e], which is the equation for the determination of S_4, is factored by S_{1y}. It follows that S_{4y} and S_4 are non-singular. I have verified that the same is true for S_6. Although I haven't attempted the proof, I am sure an inductive proof, much more concise that Poincaré's, is feasible.

Relation with the series of §125.

§211 (pp. 383-387)

In this section Poincaré extends the procedure (Delaunay's?) described in §201 to the problem involving n degrees of freedom but a single critical argument. That is, he is here interested in the passage from the classical regime to the resonance regime. Just as I was unable to reproduce his results in the earlier section, I cannot agree with his conclusions here.

Corresponding to (2) of §201, Poincaré lets

$$n_1^0 = \alpha_1^0 + \alpha_1^1 \sqrt{\mu} + \alpha_1^2 \mu + \dots \qquad (2)$$

and substitutes these formulas into the classical, non-resonant, formulation of the generating function. Then, assuming that

$$m_1 \alpha_1^0 + m_2 \alpha_2^0 + \dots + m_n \alpha_n^0 = 0,$$

he obtains a new generator S^* expanded in powers of $\sqrt{\mu}$; he writes

$$S^* = S_0' + \sqrt{\mu}\, S_1' + \mu\, S_2' + \dots \qquad (4)$$

He then derives

$$\frac{\partial S_1}{\partial y_1} = m_1 A \gamma + \frac{1}{\gamma}\frac{\partial U_1}{\partial y_1} + \frac{1}{\gamma^3}\frac{\partial U_2}{\partial y_1} + \dots \qquad (9)$$

in which A is a simple constant coefficient and

$$\gamma = m_1 \alpha_1^1 + m_2 \alpha_2^1 + \dots + m_n \alpha_n^1. \qquad (7)$$

Further, $\gamma^{-1}U_1$, $\gamma^{-3} U_2$, . . . $\gamma^{2p-1}U_p$ are analagous to $S_{1,1}$, $S_{2,1}$, ..., $S_{p,1}$ in my account of §201. While I can confirm (9), I am unable to reproduce the equation

$$\left(\frac{\partial S_1'}{\partial y_1}\right)^2 = m_1^2 A^2 \gamma^2 + m_1 A \frac{\partial U_1}{\partial y_1}, \qquad [40]$$

which compares with [4]. Poincaré asserts that the set of identities

$$2m_1 A_1 \frac{\partial U_2}{\partial y_1} + \left(\frac{\partial U_1}{\partial y_1}\right)^2 = 0 ,$$

$$2m_1 A \frac{\partial U_3}{\partial y_1} + 2 \frac{\partial U_1}{\partial y_1} \frac{\partial U_2}{\partial y_2} = 0 , \qquad [41]$$

$$\dots\dots\dots\dots$$

leads to [40]. Poincaré says of [41] that they " ... sont propriétés curieuses et inattendues ...". If these last were identities, [40] would certainly be true. However, I have not been successful in establishing the validity of [41], and thus I question [40].

As I pointed out at the end of §201, this apparent difficulty is an academic one. The passage into resonance can be made quite satisfactorily using an appropriate version of a Bohlin procedure.

Divergence of the series

§212 (pp. 388-393)

Poincaré states that the series derived in this chapter are divergent; he is clearly referring to the series for $x(=\partial S/\partial y)$ and for S. He proves this statement, for the case of circulation, by choosing a particular series of §204. Then, in view of the transformations of §206, he states that the libration case transforms to the ordinary (circulation) case; consequently the corresponding series also diverge. He defines the proof of the statement for the limiting (separatrix) case until a later chapter.

Poincaré goes on to state that the divergence is due, not to small divisors - which can be avoided as has been shown, but to the appearance of large multipliers in the numerators. For example, a term such as

$$A_p \sin(p_1 y_1 + \dots p_n y_n)$$

generates the multiplier p_i when differentiated with respect to y_i, and it is quite possible that p_i is large.

Error:

p. 393: equation (5) should read " $\dots -\sqrt{\mu}\,\dfrac{m_1\alpha_1}{(m_2\alpha_2)^2} + \dots$ "

Concluding Remarks

This Chapter has been concerned principally with the construction of the series for the generating function S in cases of resonance. His treatment refers only to problems involving a single critical argument; he does not attempt to investigate here the much more difficult problems associated with multiple resonances. He skillfully demonstrates, in the single critical argument case, how small divisors may always be avoided.

He defers until the next chapter the final stages of the solution, which involves the differentiation of S with respect to the 'new' momenta (his x_i^0) and inversion procedures. The interpretation of Poincaré's work on these matters is left to another person. For my part I am happy to rest my pen, having spent many hours, sometimes of frustration but mostly of pleasure, in my task.

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