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BOREL SUMMATION AND SPLITTING OF SEPARATRICES FOR THE HÉNON MAP

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Abstract

We study two complex invariant manifolds associated with the parabolic fixed point of the area-preserving Hénon map. A single formal power series correspond to both of them. The Borel transform of the formal series defines an analytic germ. We explore the Riemann surface and study the singularities of its analytic continuation. In particular we prove that a constant, which describes the splitting of the invariant manifolds, does not vanish.

1 Area-preserving Hénon map

One of the most interesting problems of modern dynamics is related to studying mechanisms of chaotic behavior. In Hamiltonian dynamics the splitting of separatrices is recognized to be its main source. Among the models the Hénon map plays a special role. For example the geometry of its separatrix splitting is closely related to creation of elliptic islands near a homoclinic tangency in area-preserving maps [Dua98]. The last phenomena can be modeled by the quadratic area-preserving map,

$$F_\varepsilon : (x, y) \mapsto (x_1, y_1) = (x + \varepsilon y_1,\ y + \varepsilon x(1 - x)) \ ,$$

where $\varepsilon > 0$ is a small positive parameter. It is well known that any non-trivial quadratic diffeomorphism of the plane, which preserves area and orientation and has two fixed points, can be put by a linear change of coordinates into this one-parametric family for some $\varepsilon > 0$.

The study of the separatrix splitting is especially difficult in the case of small ε, due to the exponential smallness of the splitting [FS90]. For small positive ε the origin is a hyperbolic fixed point of F_ε and the corresponding separatrices are one-dimensional curves in the plane (x, y) (see Fig. 1). The separatrices look like the separatrix of the limit flow defined by the system of two differential equations

$$\dot{x} = y \,, \qquad \dot{y} = x(1 - x) \,.$$

Unlike the separatrices of the limit flow the separatrices of the map split. The intersection of separatrices with the horizontal axis is a homoclinic point due to a symmetry of the map. It was established by one of the authors [Gel91] that the angle between the stable and unstable separatrix at the first intersection of the separatrices with the horizontal axis is given asymptotically by

$$\alpha = \frac{64\pi \mathrm{e}^{-2\pi^2/\varepsilon}}{9\varepsilon^7} \left(|\Theta| + \mathcal{O}(\varepsilon)\right) \tag{1}$$

for some $\Theta \in \mathbb{C}$. This formula implies the exponentially small transversality of the homoclinic point for all small $\varepsilon > 0$ provided the factor $|\Theta|$ does not vanish. In [Gel91] this factor was evaluated numerically: $|\Theta| \approx 2.474 \cdot 10^6$.

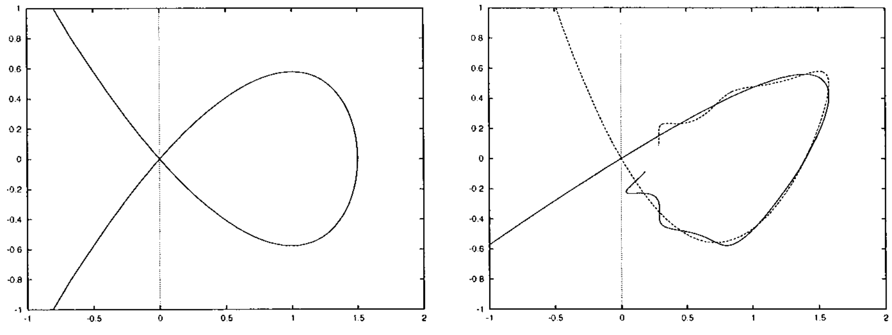

Figure 1: The limit separatrix (left) and the splitting of separatrices of the Hénon map (right). The unstable separatrix is drawn by the solid line, the stable one by the dashed line

The splitting constant Θ is a complex number, which comes from the study of the separatrices of the parabolic fixed point of the Hénon map,

$$(u, v) \mapsto (u + v - u^2, v - u^2)\,, \tag{2}$$

in the complex phase space $\mathbb{C}^2$. These separatrices are the main object of the present paper.

The study of the separatrices of the map (2) can be reduced to the study of the single second-order nonlinear finite-difference equation

$$u(z+1) - 2u(z) + u(z-1) = -u^2(z)\,, \tag{3}$$

where z is a complex variable. It is often called "complex time". Of course, this equation has a lot of solutions. The separatrix solutions are uniquely defined by the asymptotic condition

$$u^{\pm}(z) = -\frac{6}{z^2} + \mathcal{O}(z^{-4}) \qquad \text{for } z \to \pm\infty\,.$$

The sign "+" corresponds to the stable separatrix and the "−" corresponds to the unstable one. The functions $u^{\pm}$ are entire. The splitting of the separatrices may be described by the difference

$$w(z) = u^+(z) - u^-(z)\,.$$

If $\operatorname{Im} z$ goes to infinity following the negative imaginary axis this difference goes to zero exponentially fast. The splitting constant Θ describes this quantitatively:

$$w(z) = \mathrm{e}^{-2\pi \mathrm{i} z}\frac{z^4}{84}(\Theta + \mathcal{O}(z^{-2}))\,. \tag{4}$$

The last asymptotic equality can be considered as the definition of the splitting constant Θ. We discuss some alternative definitions of Θ later in Section 2.4.

Theorem 1 *In the case of the Hénon map the splitting constant* $|\Theta|$ *does not vanish. More precisely,* $\Theta \in i\mathbb{R}$ *and* $\operatorname{Im}\Theta < 0$.

The proof of Theorem 1 is based on the detailed study of the Borel transform of the formal separatrix of the parabolic fixed point. We describe the Riemann surface of the Borel transform and give a complete description of the first singularity, which contains both a polar part and an infinite order branching.

The knowledge of the singularity leads to a quite efficient and simple method for numerical evaluation of Θ (Sect. 3.2).

Many analytical phenomena described in the present paper are rather usual in Ecalle's theory of resurgent functions [Eca81]. A nice introduction to this theory can be found in the book [CNP93]. The results of the present paper look quite natural in the general context of this theory. Nevertheless, our approach is mostly elementary and does not require the knowledge of the resurgent functions theory (except for the appendix, but its result is not necessary to prove that Θ does not vanish).

Different maps can have different values of splitting constants. The methods, developed in the present paper, can be used for their study.

The first definition of a splitting constant was proposed by V.F.Lazutkin [L84, LST89] for the case of the standard map.

Hakim and Mallick [HM93] proposed to use the Borel summation for the study of the exponentially small splitting of separatrices. A more rigorous approach was used by Suris [Sur94] (for the semistandard and cubic maps), who established the relation between splitting constants and asymptotic behavior of the formal series coefficients. A modification of the last approach could lead to a proof that Θ does not vanish, but to complete the proof one still needs the knowledge of the singularity structure.

Establishing an asymptotic formula like (1) is an extremely difficult analytical problem. The first formula of this type was derived by V.F.Lazutkin [L84] for the standard map. Lazutkin's original paper was based on two conjectures, which are not proved in their complete form up to know. A complete proof of the Lazutkin asymptotic formula has been published recently in [Gel99].

A formula similar to (1) describes the splitting of a small separatrix loop, created in a saddle-center bifurcation in a general family of area-preserving maps [Gel98]. In this case each family has its own splitting constant, which is not determined by any finite jet of the functions.

2 Analytical properties of $\hat{u}$

2.1 Formal separatrix solution and its Borel transform

Let $\mathbb{C}[[z^{-1}]]$ denote the space of all formal power series in the non-positive powers of z with complex coefficients.

Lemma 1 *Any nonzero solution of Eq.* (3) *in* $\mathbb{C}[[z^{-1}]]$ *can be written in the form* $u(z+a)$, *where* $a \in \mathbb{C}$ *and* u *is the unique nonzero even solution in* $\mathbb{C}[[z^{-1}]]$,

$$u(z) = \sum_{k=1}^{\infty} \frac{a_k}{z^{2k}} = -6z^{-2} + \frac{15}{2}z^{-4} - \frac{663}{40}z^{-6} + \dots .$$

The coefficients a_k *form an alternating sequence of real numbers.*

The proof of this lemma is completely straightforward. It is sufficient to substitute the series into the equation and reexpand the left and right hand sides into power series in z^{-2}. Collecting the terms of equal order we obtain a recurrence chain of algebraic equations, which define uniquely the coefficients. We provide this recurrent rule later (see Sect. 3.2 Eq. (14)). It is useful to consider the second-order finite-difference operator in the form of an infinite-order differential operator:

$$\begin{aligned} P &= \mathrm{e}^{\partial_z} - 2 + \mathrm{e}^{-\partial_z} = 4\sinh^2\frac{\partial_z}{2} = \sum_{k=1}^{\infty} \frac{2\partial_z^{2k}}{(2k)!}, \\ (Pu)(z) &= u(z+1) - 2u(z) + u(z-1), \end{aligned}$$

where ∂_z denotes the differentiation with respect to z. □

Our main object is the formal Borel transform of u defined according to the usual rule $z^{-n-1} \mapsto \frac{\zeta^n}{n!}$, *i.e.* we study the series in positive powers of the complex variable ζ

$$\hat{u}(\zeta) = \sum_{k=1}^{\infty} a_k \frac{\zeta^{2k-1}}{(2k-1)!}. \tag{5}$$

The Borel transform converts multiplication into convolution: if $\hat{u}(\zeta)$ and $\hat{v}(\zeta)$ are formal Borel transform of the formal series $u(z)$ and $v(z)$, the transform of the product $u(z)v(z)$ is

$$(\hat{u} * \hat{v})(\zeta) = \int_0^{\zeta} \hat{u}(\zeta - \zeta')\hat{v}(\zeta')\,\mathrm{d}\zeta'.$$

The Borel transform maps the differentiation ∂_z into the operation of multiplication by $-\zeta$.

The formal Borel transform $\hat{u}$ is the unique odd solution of the convolution equation,

$$4\sinh^2\frac{\zeta}{2}\,\hat{u}(\zeta) = -(\hat{u} * \hat{u})(\zeta)\,, \tag{6}$$

in $\mathbb{C}[[\zeta]]$. In other words, this equation generates the same sequence of coefficients a_k as the original one.

Let $\mathcal{R}$ be the Riemann surface obtained by adding the origin to the first sheet of the universal covering of $\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$. It is the set of all homotopy classes (with fixed extremities) of paths issuing from 0 and lying in $\mathbb{C}\setminus 2\pi\mathrm{i}\mathbb{Z}$ (except their origin).

Theorem 2 *The formal Borel transform $\hat{u}$ is convergent at the origin and defines a holomorphic germ, which extends analytically to $\mathcal{R}$ with exponential decay at infinity on each half-sheet of $\mathcal{R}$.*

The proof of this theorem is in Sect. 4 and in the appendix. The analyticity on the main sheet of $\mathcal{R}$ was proved by V. Chernov [Che98].

In fact, $\hat{u}$ is of exponential type $-\infty$, *i.e.* it decreases faster than any exponent $\mathrm{e}^{c\zeta}$ for any real c along each nonvertical ray.

In particular, Theorem 2 implies that $\hat{u}(\zeta)$ is an odd real-analytic function in a neighborhood of the origin in the main sheet.

Theorem 3 *The analytic continuation of $\hat{u}$ to a neighborhood of its first singular point $2\pi\mathrm{i}$ in the main sheet of $\mathcal{R}$ can be written in the form,*

$$\hat{u}(\zeta) = \frac{A_5}{(\zeta-2\pi\mathrm{i})^5} + \frac{A_3}{(\zeta-2\pi\mathrm{i})^3} + \frac{A_1}{\zeta-2\pi\mathrm{i}} + h(\zeta-2\pi\mathrm{i})\frac{\log(\zeta-2\pi\mathrm{i})}{2\pi\mathrm{i}} + r(\zeta-2\pi\mathrm{i})\,, \tag{7}$$

where h and r are holomorphic at the origin and have analytical continuation onto the whole Riemann surface $\mathcal{R}$, moreover $h(0)=0$.

The proof of this theorem is in Sect. 5.

2.2 First singularity as a solution of the variational equation

We need a more detailed description of the singularity. In fact we show that the singular part of $\hat{u}(\zeta)$ at the first singularity $\zeta = 2\pi\mathrm{i}$ is described by a linear combination of two basic solutions of the variational equation near $u(z)$,

$$\varphi(z+1) - 2\varphi(z) + \varphi(z-1) = -2u(z)\varphi(z)\,. \tag{8}$$

We give the precise meaning for this statement later in this section.

First, we need some preliminary information on formal solutions of the variational equation.

Lemma 2 *The homogeneous variation equation* (8) *admits two formal solutions, which can be written in the form*

$$\varphi_1(z) = \sum_{k=1}^{\infty} \frac{b_k}{z^{2k+1}} = 12z^{-3} - 30z^{-5} + \frac{1989}{20}z^{-7} + \dots,$$

$$\varphi_2(z) = \sum_{k=-2}^{\infty} \frac{d_k}{z^{2k}} = \frac{1}{84}z^4 + \frac{17}{840}z^2 - \frac{17}{2240} + \frac{3}{35}z^{-2} + \dots.$$

The coefficients of the series are real. Any formal solution of the homogeneous variational equation[1] *can be represented as a linear combination of these two fundamental solutions.*

The proof of this lemma is comparatively simple. The first solution can be obtained by differentiation: $\varphi_1(z) = \frac{d}{dz}u(z)$. In particular, this implies $b_k = -2ka_k$, $k \geq 1$. The second solution can be found by a substitution of the series into the equation. This leads to a recurrent chain of linear algebraic equations for the coefficients d_k (the answer is explicitly written in Sect. 3.2 Eq. (16)). The coefficients d_k are defined up to a common factor. We normalized the second basic solution by

$$\mathcal{W}_{\varphi_1\varphi_2}(z) = 1,$$

where $\mathcal{W}_{\varphi_1\varphi_2}(z)$ is the finite-difference Wronskian

$$\mathcal{W}_{\varphi_1\varphi_2}(z) = \det\begin{pmatrix} \varphi_1(z-1) & \varphi_2(z-1) \\ \varphi_1(z) & \varphi_2(z) \end{pmatrix}.$$

In the theory of linear finite-difference equations its role is similar to the role of the classical Wronskian in the theory of ordinary differential equations. If $\varphi(z)$ is a solution of the homogeneous variational equation, then $c_k = \mathcal{W}_{\varphi\varphi_k}(z)$, $k = 1, 2$, are 1-periodic in z. On the other hand they are formal series and, consequently, constant. The desired representation for the solution is $\varphi(z) = c_2\varphi_1 - c_1\varphi_2$. □

The next theorem establishes a very remarkable relation between the solutions of the formal variational equation around the formal separatrix $u(z)$ and the first singularity of $\hat{u}(\zeta)$. Let us consider the formal power series

$$\varphi(z) = 2\pi \mathrm{i}\left(A_5\frac{z^4}{4!} + A_3\frac{z^2}{2!} + A_1\right) + \sum_{k=1}^{\infty}\frac{k!h_k}{z^{k+1}}, \tag{9}$$

where the infinite sum represents the formal Laplace (inverse Borel) transform of the Taylor series $h(\xi) = \sum_{k=1}^{\infty} h_k\xi^k$.

[1] In $\mathbb{C}[z][[z^{-1}]]$, the class of sums of a polynomial part plus a formal series in powers of z^{-1}

Theorem 4 *The formal power series* (9) *satisfies the variational equation* (8).

The proof of this theorem is in Sect. 6 and it is quite elementary.

Since $\varphi(z)$ is a formal solution of the variational equation (8), Lemma 2 implies that it is a linear combination of the basic solutions.

Corollary 1 *There are complex constants* μ *and* Θ *such that*

$$\varphi(z) = \Theta\varphi_2(z) + \mu\varphi_1(z)\,. \tag{10}$$

This equality may be considered as an alternative definition of the splitting constant. In Section 2.4 we prove that it is equivalent to the original definition from the introduction.

The formal equality (10) is equivalent to the following set of equalities for the coefficients:

$$A_5 = \frac{\Theta}{7\pi\mathrm{i}}\,, \qquad A_3 = \frac{17\Theta}{840\pi\mathrm{i}}\,, \qquad A_1 = -\frac{17\Theta}{4480\pi\mathrm{i}}\,,$$

$$h_{2k-1} = \Theta\,\frac{d_k}{(2k-1)!}\,, \qquad h_{2k} = \mu\,\frac{b_k}{(2k)!}\,, \qquad \text{for all } k \geq 1, \tag{11}$$

where b_k and d_k are defined by Lemma 2 as the coefficients of the formal solutions $\varphi_1(z)$ and $\varphi_2(z)$. In particular $h_1 = \frac{3\Theta}{35}$ and $h_2 = 6\mu$.

Remark. Theorem 4 is almost trivial from the viewpoint of Resurgence theory. Indeed in terms of this theory $\varphi = \Delta_{2\pi\mathrm{i}}u$, where $\Delta_{2\pi\mathrm{i}}$ is an alien derivation. The theorem follows immediately from Eq. (3) since the operator $\Delta_{2\pi\mathrm{i}}$ commutes with translations of step 1 and obeys the Leibniz rule.

2.3 The splitting constant Θ doesn't vanish

Proposition 1 *The constant* Θ *is purely imaginary,* $\operatorname{Im}\Theta < 0$, *and* μ *is real.*

Proof. Since the function $\hat{u}$ is real-analytic and odd, it is purely imaginary on the imaginary axis. Consequently, the coefficients A_5, A_3, A_1 are real, the h_{2k-1} are purely imaginary and the h_{2k} are real. This implies that Θ is purely imaginary and μ is real.

Suppose $\Theta = 0$. This leads to a contradiction. Indeed in this case $A_5 = A_3 = A_1 = 0$ due to Corollary 1 and it follows from Theorem 3 that $\hat{u}$ would be bounded on the segment $[0, 2\pi\mathrm{i}]$. The following trivial arguments show that this is impossible. Let $\hat{v}(y) = \mathrm{i}\hat{u}(\mathrm{i}y)$, it has the Taylor expansion $\hat{v}(y) = \sum_{k=1}^{\infty}(-1)^k \frac{a_k y^{2k-1}}{(2k-1)!}$, where all the coefficients are positive. The radius of convergence of the series is exactly 2π. If the supposition were true the real-analytic positive function $\hat{v}$ would be bounded on the interval $[0, 2\pi)$. But it satisfies the convolution equation

$$4\sin^2\frac{y}{2}\;\hat{v}(y) = (\hat{v} * \hat{v})(y)\,,$$

which can be easily derived from Eq. (6). If $\hat{v}$ were bounded, the left-hand side of the last equation would converge to zero when $y \to 2\pi$. This is impossible, since the right-hand side is an integral of a positive function.

Since the function $\hat{v}(y)$ is positive and $\hat{u}(\mathrm{i}y) = -\mathrm{i}\hat{v}(y)$, A_5 is negative. This implies that $\operatorname{Im}\Theta = 7\pi A_5 < 0$. □

2.4 Splitting of complex separatrices

Now we are ready to check that the constant Θ, defined in the previous section, is the splitting constant in the sense of the definition from the introduction The Laplace integrals

$$u^{\pm}(z) = \int_0^{\pm\infty} \mathrm{e}^{-z\zeta}\hat{u}(\zeta)\,\mathrm{d}\zeta$$

define two entire functions which satisfy Eq. (3). (In some sense—be careful about the domains—the Laplace transform is inverse to the Borel transform.) These functions—together with the functions $v^{\pm}(z) = u^{\pm}(z) - u^{\pm}(z-1)$—represent the stable and unstable separatrices of the parabolic fixed point of the Hénon map (2). We study the splitting of these complex separatrices, which is described by the function

$$w(z) = u^{+}(z) - u^{-}(z) = \int_{-\infty}^{\infty} \mathrm{e}^{-z\zeta}\hat{u}(\zeta)\,\mathrm{d}\zeta\,.$$

The integral is taken along the real axis on the main sheet of $\mathcal{R}$.

Lemma 3 *For large negative* $\operatorname{Im} z$ *there is an asymptotic equality,*

$$w(z) \overset{\mathrm{as}}{=} (\mu\varphi_1(z) + \Theta\varphi_2(z))\,\mathrm{e}^{-2\pi\mathrm{i}z}\,.$$

The symbol $\overset{\mathrm{as}}{=}$ indicates that the relation is asymptotic, *i.e.* φ_1 and φ_2 are formal series; if one retains only a finite number of terms, then the error would be of the order of the first missing term.

To prove this lemma it is sufficient to deform the path into the upper half-plane, and compute the contribution of the first singularity. Note that a part of the path goes to the second sheet of the Riemann surface $\mathcal{R}$ (see Fig. 2). □

Since $\varphi_2(z) = z^4/84 + \mathcal{O}(z^2)$ and $\varphi_1 = \mathcal{O}(z^{-3})$, the lemma implies the estimate (4). Thus the constant Θ of the previous section is actually the splitting constant.

The first definition of the splitting constant was proposed by V.F.Lazutkin [L84, LST89] for the case of the standard map. The constant was defined as a first-order Fourier coefficient of an "energy" splitting function, $\mathcal{E}(u^{+}(z)) - \mathcal{E}(u^{-}(z))$. This definition was used by one of the authors, who obtained the

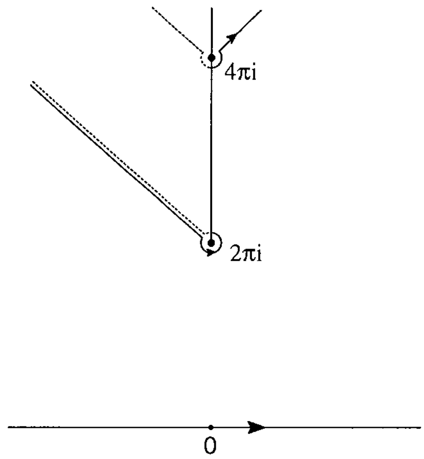

Figure 2: The integration path is moved into the upper half of the first sheet of the Riemann surface. Then it is deformed and pushed to the second sheet through the cut between the singularities at 2πi and 4πi

numerical value, $|\Theta| \approx 2.474 \cdot 10^6$ [GLT91, Gel91]. The last two papers also contain the values of the splitting constant for the cubic and some other polynomial area-preserving maps.

In [GLS94] it was shown that this definition is equivalent to the following one:

$$\begin{aligned} \Theta &= \lim_{\operatorname{Im} z \to -\infty} \mathrm{e}^{2\pi i z} \mathcal{W}_{w\varphi_2^-}(z) \,, \\ \mu &= \lim_{\operatorname{Im} z \to -\infty} \mathrm{e}^{2\pi i z} \mathcal{W}_{\varphi_1^- w}(z) \,, \end{aligned}$$

where $\mathcal{W}$ is the finite-difference Wronskian. The limit is reached exponentially fast due to the following estimate:

$$w(z) = \left(\mu\varphi_1^-(z) + \Theta\varphi_2^-(z)\right) \mathrm{e}^{-2\pi i z} + \mathcal{O}(z^{10}\mathrm{e}^{-4\pi i z}),$$

which is written here for the Hénon map, in terms of φ_1^- and φ_2^- which are the Borel-Laplace sums of φ_1 and φ_2 corresponding to Laplace integration over $\mathbb{R}^-$. The computations, based on this definition, afford to compute between 6 and 8 correct decimals of Θ using the double precision complex arithmetic.

In the next section we will prove that in the case of the Hénon map

$$|\Theta| = 42 \lim_{k\to\infty} \frac{(-1)^{k+1} a_k (2\pi)^{2k+5}}{(2k+3)!} \,.$$

This formula, established by semi-empirical reasoning, was used by several authors [Che98], [TTJ98] for numerical evaluation of the splitting constant. The computation of V. Chernov [Che98] (14 correct decimals) are in excellent agreement with the present paper as well as with [Gel91, GLT91], where an independent method was used. On the other hand we are not able to explain the discrepancy with the numerical experiments of Tovbis et al. [TTJ98], where for the constant $K = |\Theta|/168\pi$ it was obtained $K \approx 7374$, which is some 36% larger than we expect.

3 Evaluation of the splitting constants

3.1 Complex singularities and asymptotic behavior of Taylor coefficients

It is well known that an analytic function is completely defined by its germ in a neighborhood of any point of its domain. In particular, this implies that the Taylor series of $\hat{u}$ at the origin contains information about all the singularities. Here we show how the information about the first singularities can be extracted from the asymptotic behavior of the Taylor coefficients. In the next section we show that this leads to a highly efficient numerical method for evaluation of the splitting constants Θ and μ.

The function $\hat{u}(\zeta)$ has singularities on the boundary of $D_{2\pi} = \{\, \zeta \in \mathbb{C} : |\zeta| < 2\pi \,\}$ at $\pm 2\pi \mathrm{i}$. Let g be the corresponding polar part:

$$
\begin{aligned}
g(\zeta) &= \frac{A_5}{(\zeta - 2\pi\mathrm{i})^5} + \frac{A_3}{(\zeta - 2\pi\mathrm{i})^3} + \frac{A_1}{\zeta - 2\pi\mathrm{i}} \\
&\quad + \frac{A_5}{(\zeta + 2\pi\mathrm{i})^5} + \frac{A_3}{(\zeta + 2\pi\mathrm{i})^3} + \frac{A_1}{\zeta + 2\pi\mathrm{i}} \\
&= \left(\frac{A_5}{4!} \partial_\zeta^4 + \frac{A_3}{2!} \partial_\zeta^2 + A_1 \right) \frac{2\zeta}{4\pi^2 + \zeta^2} ,
\end{aligned}
$$

where we used the symmetries of the singularities due to the fact that $\hat{u}$ is real-analytic and odd. Since

$$
\frac{2\zeta}{4\pi^2 + \zeta^2} = \frac{1}{\pi} \sum_{k=1}^{\infty} (-1)^{k-1} \left(\frac{\zeta}{2\pi} \right)^{2k-1} ,
$$

we obtain

$$
\begin{aligned}
\partial_\zeta^{2k-1} g(0) &= \frac{A_5}{4!\pi} \frac{(-1)^{k-1}(2k+3)!}{(2\pi)^{2k+3}} \\
&\quad + \frac{A_3}{2!\pi} \frac{(-1)^{k-1}(2k+1)!}{(2\pi)^{2k+1}} + \frac{A_1}{\pi} \frac{(-1)^{k-1}(2k-1)!}{(2\pi)^{2k-1}} . \qquad (12)
\end{aligned}
$$

All the derivatives of even order vanish at zero.

The difference $f(\zeta) = \hat{u}(\zeta) - g(\zeta)$ is analytic in $D_{2\pi}$ and continuous in its closure. Applying the Cauchy estimates

$$|\partial_\zeta^k f(0)| \le \frac{k!}{(2\pi)^k} \sup_{\zeta \in D_{2\pi}} |f(\zeta)|$$

and taking into account that $\partial_\zeta^{2k-1}\hat{u}(0) = a_k$, we see that

$$a_k - \partial_\zeta^{2k-1} g(0) = \mathcal{O}\left(\frac{(2k-1)!}{(2\pi)^{2k-1}}\right). \tag{13}$$

If we keep only the first term in the expression for $\partial_\zeta^{2k-1} g(0)$ and solve the equation with respect to A_5, we obtain

$$A_5 = \frac{4!}{2} \frac{(-1)^{k-1} a_k (2\pi)^{2k+4}}{(2k+3)!} + \mathcal{O}(k^{-2}).$$

It is not too difficult to compute several hundreds of a_k. This formula was used by different authors to compute A_5 and, consequently, evaluate $\Theta = 7\pi \mathrm{i} A_5$. The convergence of the method is rather slow. We can substantially improve it using more detailed knowledge of the singularity structure. As a first improvement we note that $A_3 = 17A_5/120$. Then we substitute the first two terms of (12) into (13) and solve the equation with respect to A_5:

$$A_5 = \frac{(-1)^{k-1} a_k (2\pi)^{2k+4}}{(2k+3)!} \left(\frac{2}{4!} + \frac{17}{120} \frac{(2\pi)^2}{(2k+2)(2k+3)}\right)^{-1} + \mathcal{O}(k^{-4}).$$

In this way we constructed a sequence, which converges to A_5 much faster.

If we repeat the same reasoning adding to g the first N, $N \ge 1$, terms of the logarithmic part of the singularities at $\pm 2\pi\mathrm{i}$, *i.e.* if instead of g we consider

$$g_N(\zeta) = s_N(\zeta - 2\pi\mathrm{i}) + \overline{s_N(\overline{\zeta} - 2\pi\mathrm{i})},$$

where

$$s_N(\xi) = \frac{A_5}{\xi^5} + \frac{A_3}{\xi^3} + \frac{A_1}{\xi} + \sum_{m=1}^{N} h_m \xi^m \frac{\log \xi}{2\pi\mathrm{i}},$$

we obtain an approximation for a_k with relative error $\mathcal{O}(k^{-N-7})$. This approximation contains h_k, for which Corollary 1 establishes the relation to the splitting constants Θ and μ.

Of course, the constant in the $\mathcal{O}$ estimates depends on N. In the next section we numerically observe that choosing $N = k/2$ leads to exponential convergence (relative error is $\mathcal{O}(\mathrm{e}^{-ck})$). From the theoretical point of view this is quite natural and can be rigorously proved by applying the techniques of the present paper. This is due to the fact that φ_1 and φ_2 are resurgent functions too, which provides Gevrey-1 estimates on the growth of the sequences b_k, d_k and the constants in the $\mathcal{O}$-terms.

3.2 Numerical algorithm

Using the complete knowledge on the singularities structure, we construct a very efficient numerical method for the evaluation of the splitting constants Θ and μ. The algorithm is extremely simple. It is based on the arguments of the previous section.

First, we compute the coefficients of u and of φ_1, the first formal solution of the variational equation. We use the following recurrent formulas. We let $a_1 = -6$, $a_2 = 15/2$, and

$$a_m = -\frac{1}{2m^2+m-6}\left(\sum_{k=1}^{m-1}\binom{2m+1}{2k-1}a_k + \sum_{k=2}^{m-1}\frac{a_{m+1-k}a_k}{2}\right) \quad \text{for } m \geq 3, \tag{14}$$

$$b_m = -2m\, a_m \quad \text{for } m \geq 1. \tag{15}$$

Then we compute the coefficients of φ_2, the second solution of the variational equation. We let $d_{-2} = 1/84$, $d_{-1} = 17/84$, $d_0 = -17/2240$, and

$$d_m = -\frac{1}{2m^2+m-6}\left(\sum_{k=1}^{m-1}\binom{2m+1}{2k-1}d_k + \sum_{k=-2}^{m-1} a_{m+1-k}d_k\right), \quad m \geq 1. \tag{16}$$

Then we evaluate two auxiliary sums:

$$s^1_{k,N} = \sum_{m=-2}^{N-1}\frac{2(-1)^l(2l+1)!d_m}{(2\pi)^{2l+3}}, \qquad (l = k-m-1)$$

$$s^2_{k,N} = \sum_{m=1}^{N-1}\frac{2(-1)^l(2l)!b_m}{(2\pi)^{2l+2}}, \qquad (l = k-m-1).$$

Finally, we evaluate the splitting constants Θ and μ by comparing a_k with

$$\tilde{a}_k = s^1_{k,N}\Theta + s^2_{k,N}\mu, \tag{17}$$

the k^{th} derivative of $g_N(\zeta)$ at origin (provided $k \geq 2N$). According to the previous section $a_k = \tilde{a}_k(1+O(k^{-N-7}))$. In our experiments we used $N = k/2$, which seems to minimize the error due to the replacement of $\tilde{a}_k$ by a_k. In fact our numerical method is not sensitive to this choice. The method of the present paper can be used to analyze this error analytically.

In the numerical experiments we computed $n = 50$ terms for each of the sequences $(a_k,\, b_k,\, d_k,\, s^1_{k,N},\, s^2_{k,N})$. Then we found the constants,

$$\Theta^* = 2.474\,425\,593\,553\,251\,053\,840 \cdot 10^6,$$
$$\mu^* = 4\,908.934\,252\,164,$$

by replacing $\tilde{a}_k$ by a_k and solving (17) by the method of least squares using the last six values of $k = 45 \dots 50$.

In order to estimate the error due to the replacement of $\tilde{a}_k$ by a_k we computed the relative errors

$$\delta_k = \frac{\tilde{a}_k^* - a_k}{a_k},$$

where $\tilde{a}_k^* = s_{k,N}^1 \Theta^* + s_{k,N}^2 \mu^*$ is the "experimental" value of $\tilde{a}_k$. Some particular values are

$$\delta_{10} \approx 4 \cdot 10^{-3}, \qquad \delta_{20} \approx -7 \cdot 10^{-7}, \qquad \delta_{30} \approx -4 \cdot 10^{-13}.$$

The result shown on Fig. 3 gives a numerical evidence of $\delta_k \sim e^{-ck}$. From the analytical viewpoint this error is due to a contribution from the other singularities of $\hat{u}$.

Finally, we repeated the computations for larger values of n. We used the coefficients with $k = 90 \dots 100$ to determine the values of Θ and μ. This test confirmed that the previously computed decimals are all correct.

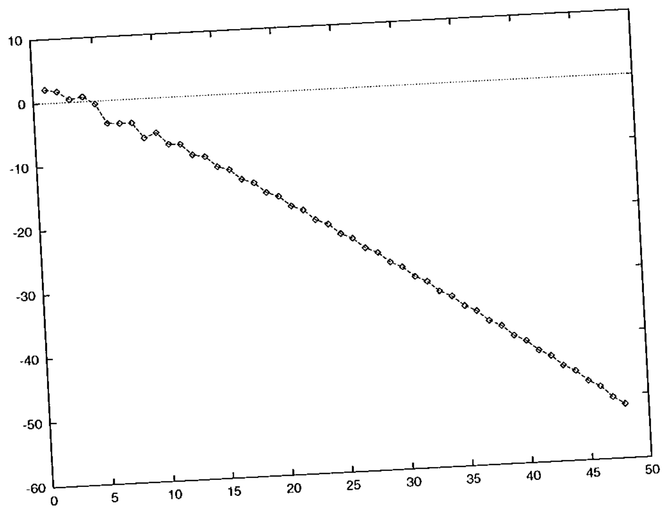

Figure 3: The plot of $\log|\delta_k|$ versus k

4 Proof of the analyticity of $\hat{u}$ on the first sheets

Theorem 2 claims that $\hat{u}$ extends analytically to $\mathcal{R}$. In this section we only prove that $\hat{u}$ extends analytically on the main sheet of $\mathcal{R}$ and on the first half-sheets, *i.e.* the half-sheets which can be reached from the main sheet by crossing the imaginary axis exactly once. This weaker form of the theorem is sufficient for the rest of the paper. Moreover its proof is elementary. The complete proof of Theorem 2 in the whole requires some knowledge of the Resurgence theory and it is sketched together with the explanations of the basic notions in the appendix.

Let us recall some definitions first. The Riemann surface $\mathcal{R}$ is obtained by adding the origin to the main sheet of the universal covering of $\mathbb{C} \setminus 2\pi i\mathbb{Z}$. It is also the set of all homotopy classes[2] of paths issuing from 0 and lying in $\mathbb{C}\setminus 2\pi i\mathbb{Z}$ (except for their origin). The natural projection $\zeta \in \mathcal{R} \mapsto \dot{\zeta} \in (\mathbb{C} \setminus 2\pi i\mathbb{Z}) \cup \{0\}$ ($\dot{\zeta}$ is the extremity of any path representing ζ) is locally biholomorphic in a neighborhood of every point.

Let us introduce two open subsets $\mathcal{R}^{(0)} \subset \mathcal{R}^{(1)}$ of the Riemann surface $\mathcal{R}$. We will denote by $\mathcal{R}^{(0)}$ the set $\mathbb{C} \setminus \pm 2\pi i[1, +\infty[$ (the complex plane deprived from the two singular half-lines $2\pi i[1, +\infty[$ and $-2\pi i[1, +\infty[$), which can be identified with the set of homotopy classes of paths issuing from 0 and lying in $\mathbb{C} \setminus \pm 2\pi i[1, +\infty[$. This is the "main sheet" of $\mathcal{R}$. The union of the "nearby half-sheets" will be denoted by $\mathcal{R}^{(1)}$: it is the set of homotopy classes of paths issuing from 0, lying in $\mathbb{C} \setminus 2\pi i\mathbb{Z}$ and crossing the imaginary axis at most once. We arrive to a nearby half-sheet, when we follow a path which crosses the imaginary axis between two singular points. We arrive to different sheets of the Riemann surface, when we pass between different singularities. Thus there are infinitely many nearby half-sheets.

For a given analytic germ at the origin, saying that it extends to an analytic function on $\mathcal{R}$ (*resp.* $\mathcal{R}^{(0)}$, *resp.* $\mathcal{R}^{(1)}$) amounts to saying that any path which represents an element of $\mathcal{R}$ (*resp.* $\mathcal{R}^{(0)}$, *resp.* $\mathcal{R}^{(1)}$) is a path of analytic continuation for it.

Theorem 5 *The formal Borel transform $\hat{u}$ is convergent at the origin and defines a holomorphic germ, which extends analytically to $\mathcal{R}^{(1)}$ with exponential decay at infinity on each half-sheet of $\mathcal{R}^{(1)}$.*

The rest of the section contains the proof of this theorem. This is a necessary step for proving Theorem 2 completely.

Our goal is to prove that the series $\hat{u}(\zeta) \in \mathbb{C}[[\zeta]]$, defined by the formula (5) as the formal Borel transform of the unique nonzero even solution of (3), has a nonzero radius of convergence and to follow the analytic continuation of the corresponding holomorphic germ. We claimed that the disk of convergence was $D_{2\pi}$, but the inductive computation of its coefficients a_k does not help much in the study of the analytic continuation outside this disk.

This is why we use an alternative representation of $\hat{u}$, expressing it as the limit of some iterative scheme at each step of which properties of analyticity can

[2] When speaking of homotopy of paths, we always refer to homotopy with fixed extremities.

be checked in $\mathcal{R}^{(1)}$. Then the theorem follows from the uniform convergence of the scheme (on a system of subsets, the union of which covers $\mathcal{R}^{(1)}$).

We first define the new unknown series $\hat{v}$ by

$$\hat{u}(\zeta) = -6\zeta + \hat{v}(\zeta).$$

In the sequel we only deal with $\hat{v}$, which is the unique solution in $\zeta^3\mathbb{C}[[\zeta]]$ of the convolution equation

$$\alpha\hat{v} - 12\zeta * \hat{v} = \hat{w}_0 - \hat{v} * \hat{v}, \tag{18}$$

where $\alpha(\zeta) = 4\sinh^2\frac{\zeta}{2}$ and $\hat{w}_0(\zeta) = 6[\zeta\alpha(\zeta) - \zeta^3]$.

4.1 Iterative scheme

Let us introduce some auxiliary meromorphic functions:

$$\begin{aligned}
\beta(\zeta) &= \frac{1}{\alpha(\zeta)} \in \zeta^{-2}\mathbb{C}\{\zeta\},\\
Y(\zeta) &= \frac{3\cosh\frac{\zeta}{2}(3 + 2\cosh^2\frac{\zeta}{2})}{2\sinh^5\frac{\zeta}{2}} \in \zeta^{-5}\mathbb{C}\{\zeta\},\\
Z(\zeta) &= -\zeta Y(\zeta) + \frac{4 + 11\cosh^2\frac{\zeta}{2}}{\sinh^4\frac{\zeta}{2}} \in \zeta^2\mathbb{C}\{\zeta\}.
\end{aligned}$$

We will denote by $\int_0 f$ the formal series or the function $\zeta \mapsto \int_0^\zeta f(\zeta_1)\,\mathrm{d}\zeta_1$, whenever f is a formal series ($f \in \mathbb{C}[[\zeta]]$) or a holomorphic function ($f \in \mathbb{C}\{\zeta\}$).

Lemma 4 (The operator E) *The operator*

$$\hat{V} \in \zeta^3\mathbb{C}[[\zeta]] \mapsto \hat{W} = \alpha\hat{V} - 12\zeta * \hat{V} \in \zeta^5\mathbb{C}[[\zeta]]$$

is invertible and its inverse E can be expressed as

$$\hat{W} \in \zeta^5\mathbb{C}[[\zeta]] \mapsto E.\hat{W} = \beta\hat{W} + \frac{1}{12}Z\int_0 Y\hat{W} - \frac{1}{12}Y\int_0 Z\hat{W}.$$

If $\hat{W} \in \zeta^n\mathbb{C}[[\zeta]]$ with $n \geq 5$, $E.\hat{W} \in \zeta^{n-2}\mathbb{C}[[\zeta]]$.

If $\hat{W} \in \zeta^5\mathbb{C}\{\zeta\}$ and if the germ defined by $\hat{W}$ extends analytically to $\mathcal{R}^{(1)}$ (resp. *to $\mathcal{R}$), $E.\hat{W} \in \zeta^3\mathbb{C}\{\zeta\}$ and the germ defined by $E.\hat{W}$ extends analytically to $\mathcal{R}^{(1)}$ too* (resp. *to $\mathcal{R}$).*

Proof. Let $\hat{W} \in \zeta^5\mathbb{C}[[\zeta]]$. In order to find $\hat{V}$, we use the change of unknown function $\hat{V} = \beta F$ and we differentiate twice the operator that we want to invert: $\hat{V} \in \zeta^3\mathbb{C}[[\zeta]]$ is solution of

$$\alpha\hat{V} - 12\zeta * \hat{V} = \hat{W}$$

if and only if

$$F = \alpha \hat{V} \in \zeta^5 \mathbb{C}[[\zeta]] \qquad \text{and} \qquad F'' - 12\beta F = \hat{W}''. \tag{19}$$

One checks easily that $y = \alpha Y/12 \in \zeta^{-3}\mathbb{C}\{\zeta\}$ and $z = \alpha Z/12 \in \zeta^4 \mathbb{C}\{\zeta\}$ are independent solutions of the corresponding homogeneous equation $f'' - 12\beta f =$ 0, with Wronskian $yz' - y'z = 1$ (in fact $z = y \int_0 y^{-2}$). Thus, whenever $g \in$ $\zeta^3\mathbb{C}[[\zeta]]$, the solutions of $f'' - 12\beta f = g$ are the series $f = -y \int_0 zg + z \int_0 yg +$ $c_1 y + c_2 g$, and among them only $f = -y \int_0 zg + z \int_0 yg$ lies in $\zeta^5\mathbb{C}[[\zeta]]$.

Hence a unique solution for (19):

$$F = -y \int_0 z\hat{W}'' + z \int_0 y\hat{W}'' = -y \int_0 Z\hat{W} + z \int_0 Y\hat{W} + \hat{W}$$

(the last identity stems from a double integration by part). Multiplying by β, we obtain the desired formula for $\hat{V}$.

The property of decreasing the valuation by 2 at most is easily checked.

If $\hat{W}$ is a convergent power-series, so is $\hat{V}$. The analyticity in $\mathcal{R}^{(1)}$ or $\mathcal{R}$ is preserved because Y and Z are meromorphic with poles in $2\pi i\mathbb{Z}$ only. □

Lemma 5 (Algorithm for the $\hat{v}_n'$s) *The formulas*

- $\hat{w}_0 = 6[\zeta\alpha(\zeta) - \zeta^3] \in \zeta^5\mathbb{C}[[\zeta]]$,
- $\hat{v}_n = E.\hat{w}_n, \qquad n \geq 0,$
- $\hat{w}_n = -\displaystyle\sum_{n_1+n_2=n-1} \hat{v}_{n_1} * \hat{v}_{n_2}, \qquad n \geq 1,$

define inductively two sequences of formal series satisfying

$$\forall n \geq 0, \quad \hat{v}_n \in \zeta^{2n+3}\mathbb{C}[[\zeta]], \quad \hat{w}_n \in \zeta^{2n+5}\mathbb{C}[[\zeta]],$$

and such that the unique nonzero odd solution of (6) is

$$\hat{u}(\zeta) = -6\zeta + \sum_{n\geq 0} \hat{v}_n(\zeta).$$

Proof. The properties of the operator E ensure that the series $\hat{u}_n$ and $\hat{v}_n$ are well defined by induction, with valuations bounded from below as indicated in Lemma 5. Thus the series of formal series

$$\hat{v} = \sum_{n\geq 0} \hat{v}_n \quad \text{and} \quad \hat{w} = \sum_{n \geq 0} \hat{w}_n$$

are convergent in $\mathbb{C}[[\zeta]]$. We have

$$\hat{u} \in \zeta^3\mathbb{C}[[\zeta]], \quad \alpha\hat{v} - 12\zeta * \hat{v} = \hat{w} \qquad \text{and} \qquad \hat{w} = \hat{w}_0 - \hat{v} * \hat{v}$$

by construction, hence the result. □

It is a well-known result of Resurgence theory that, if two germs extend analytically to $\mathcal{R}$, their convolution product has the same property. (We will recall the reason why this is so in Section 4.3.) This fact and the last part of Lemma 4 show that each power-series $\hat{v}_n$ or $\hat{w}_n$ has nonzero radius of convergence and defines a germ which extends analytically to $\mathcal{R}$, since we start with $\hat{w}_0$ which converges to an entire function. We won't try to prove the convergence of the series $\sum \hat{v}_n$ in the whole Riemann surface $\mathcal{R}$ now, but we retain that each term extends analytically to $\mathcal{R}^{(1)}$.

In order to prove Theorem 5, it is thus sufficient to study the convergence of $\sum \hat{v}_n$ as a series of holomorphic functions in $\mathcal{R}^{(1)}$. We will begin by restricting ourselves to the main sheet $\mathcal{R}^{(0)}$, *i.e.* to the holomorphic star of these functions.

4.2 Convergence on the main sheet.

For $\rho \in]0, \pi/2[$, we define $\mathcal{D}_\rho$ to be a closed subset of $\mathbb{C}$ obtained by removing the open disks of center $\pm 2\pi\mathrm{i}$ and radius ρ and all the points which are "hidden" by those disks from an observer based at the origin:

$$\mathcal{D}_\rho = \mathbb{C} \setminus \{ t\zeta,\ t \in]1, +\infty[,\ \zeta \in D(\pm 2\pi\mathrm{i}, \rho) \}.$$

The main sheet of $\mathcal{R}$ obviously coincides with the union of all these sets.

Lemma 6 (Initial bounds) *For any $\rho \in]0, \pi/2[$, there exist positive numbers c, c_0 such that*

$$\forall \zeta \in \mathcal{D}_\rho \setminus \{0\}, \quad \begin{cases} |\beta(\zeta)| & \le c^2 |\zeta|^{-2}, \\ |Y(\zeta)| & \le c|\zeta|^{-5}, \\ |Z(\zeta)| & \le c|\zeta|^2, \end{cases}$$

and

$$\forall \zeta \in \mathcal{D}_\rho, \quad \hat{v}_0(\zeta) \le c_0 \frac{|\zeta|^3}{3!}.$$

Proof. Let $\rho \in]0, \pi/2[$. We observe that $\forall \zeta \in \mathcal{D}_\rho$, $\operatorname{Re}\zeta \ge \frac{\rho}{\sqrt{\rho^2+4\pi^2}}|\zeta|$. Let us first consider the functions β, Y and Z: they are analytic in $\mathcal{D}_\rho$, except at the origin for β and Y which have poles of order 2 and 5 there, whereas Z has a zero of order 2 at the origin. On the other hand these functions decay exponentially when $|\zeta|$ tends to infinity (with ζ remaining in $\mathcal{D}_\rho$), because

$$\lim_{\operatorname{Re}\zeta \to \pm\infty} \mathrm{e}^{\pm\zeta}\, \beta(\zeta) = 1,$$

$$\lim_{\operatorname{Re}\zeta \to \pm\infty} \mathrm{e}^{\pm\zeta}\, Y(\zeta) = 12,$$

$$\lim_{\operatorname{Re}\zeta \to \pm\infty} \zeta^{-1}\, \mathrm{e}^{\pm\zeta}\, Z(\zeta) = -12,$$

and exponential decay with respect to $|\operatorname{Re}\zeta|$ in $\mathcal{D}_\rho$ means exponential decay with respect to $|\zeta|$, hence the result.

Now $\hat{v}_0 = \beta\hat{w}_0 + \frac{1}{12}Z\int_0 Y\hat{w}_0 - \frac{1}{12}Y\int_0 Z\hat{w}_0$, where $\hat{w}_0$ is an entire function of order 1 satisfying

$$\lim_{\operatorname{Re}\zeta\to\pm\infty} \zeta^{-1}\,\mathrm{e}^{-(\pm\zeta)}\,\hat{w}_0(\zeta) = 6.$$

Thus $\hat{w}_0(\zeta) \leq \text{const}\,|\zeta|\,\mathrm{e}^{|\operatorname{Re}\zeta|}$ for $|\zeta| > \rho$. On the other hand $\hat{w}_0(\zeta) = \mathcal{O}(\zeta^5)$ near the origin. From that we deduce inequalities

$$|Y\hat{w}_0| \leq \text{const}\,(1+|\zeta|) \quad\text{and}\quad |Z\hat{w}_0| \leq \text{const}\,(|\zeta|^2+|\zeta|^7) \quad \text{in } \mathcal{D}_\rho$$

which show that $\hat{v}_0(\zeta) \leq \text{const}\,|\zeta|^3$ for $\zeta > \rho$. And the proof is complete since $\hat{v}_0(\zeta) = \mathcal{O}(\zeta^3)$. □

Lemma 7 (Bounds in the main sheet) *Let $\rho \in]0,\pi/2[$.*
(a) If $\hat{F}$ and $\hat{G}$ are holomorphic functions in $\mathcal{D}_\rho$ which satisfy

$$\forall\zeta\in\mathcal{D}_\rho,\quad |\hat{F}(\zeta)| \leq \mathcal{F}(|\zeta|) \quad\textit{and}\quad |\hat{G}(\zeta)| \leq \mathcal{G}(|\zeta|),$$

where $\mathcal{F}$ and $\mathcal{G}$ are continuous functions on $\mathbb{R}^+$, their convolution product $\hat{F}*\hat{G}$ is holomorphic in $\mathcal{D}_\rho$ and satisfies

$$\forall\zeta\in\mathcal{D}_\rho,\quad |(\hat{F}*\hat{G})(\zeta)| \leq (\mathcal{F}*\mathcal{G})(|\zeta|).$$

(b) If $\hat{W}$ is holomorphic in $\mathcal{R}_\rho$ and satisfies

$$\forall\zeta\in\mathcal{D}_\rho,\quad |\hat{W}(\zeta)| \leq C|\zeta|^\nu$$

for some real $C>0$ and integer $\nu\geq 5$, the function $E.\hat{W}$ is holomorphic in $\mathcal{D}_\rho$ and satisfies

$$\forall\zeta\in\mathcal{D}_\rho,\quad |(E.\hat{W})(\zeta)| \leq 2c^2C|\zeta|^{\nu-2}$$

with c as in Lemma 6.

Proof. Part *(a)* is quite obvious since $(\hat{F}*\hat{G})(\zeta) = \zeta\int_0^1 \hat{F}(t\zeta)\hat{G}((1-t)\zeta)\,\mathrm{d}t$.

Let $\rho\in]0,\pi/2[$ and $\hat{W}$, C, ν as in Part *(b)*. For $\zeta\in\mathcal{D}_\rho$, we can write $(E.\hat{W})(\zeta)$ as the sum of three terms:

$$(E.\hat{W})(\zeta) = \beta(\zeta)\hat{W}(\zeta) + \frac{Z(\zeta)}{12}\int_0^1 \zeta Y(t\zeta)\hat{W}(t\zeta)\,\mathrm{d}t - \frac{Y(\zeta)}{12}\int_0^1 \zeta Z(t\zeta)\hat{W}(t\zeta)\,\mathrm{d}t.$$

By virtue of the previous lemma, the first term is bounded by $c^2C|\zeta|^{\nu-2}$, the second one by $\frac{c^2C}{12(\nu-4)}|\zeta|^{\nu-2}$ and the third one by $\frac{c^2C}{12(\nu+3)}|\zeta|^{\nu-2}$, hence the result. □

Lemma 8 (Convergence in the main sheet) *Let $\rho \in]0, \pi/2[$ and $c, c_0 > 0$ as in Lemma 6. The formulas*

$$c'_n = \sum_{n_1+n_2=n-1} c_{n_1} c_{n_2}, \quad c_n = \frac{c^2}{21} c'_n, \qquad n \geq 1$$

define inductively two sequences of positive numbers satisfying

$$\forall \zeta \in \mathcal{D}_\rho, \quad |\hat{v}_n(\zeta)| \leq c_n \frac{|\zeta|^{2n+3}}{(2n+3)!} \quad \text{and} \quad |\hat{w}_n(\zeta)| \leq c'_n \frac{|\zeta|^{2n+5}}{(2n+5)!}.$$

The series of functions $\sum \hat{v}_n$ converges uniformly in $\mathcal{D}_\rho$ to a holomorphic function $\hat{v}$ and $\hat{u} = -6\zeta + \hat{v}$ has exponential decay at infinity in $\mathcal{D}_\rho$.

Proof. Taking into account the bound for v_0 which is provided by Lemma 6, we proceed by induction and suppose that $\hat{v}_0, \ldots, \hat{v}_{n-1}$ are bounded as indicated in Lemma 8 for some $n \geq 1$. The desired bound for $\hat{w}_n$ is obtained by Part (a) of Lemma 7, since

$$n_1 + n_2 = n - 1 \quad \Rightarrow \quad \frac{\zeta^{2n_1+3}}{(2n_1+3)!} * \frac{\zeta^{2n_2+3}}{(2n_2+3)!} = \frac{\zeta^{2n+5}}{(2n+5)!}.$$

Then we derive the bound for $\hat{v}_n$ by Part (b) of Lemma 7, since $(2n+4)(2n+5) \geq 42$.

Let $\lambda = 4c^2/21$. The generating series $c(X) = \sum_{n\geq 0} c_n X^n$ is easily computed: $c(X) = c_0 + \frac{\lambda}{4} X c(X)^2$, thus

$$c(X) = 2\frac{1 - (1 - c_0\lambda X)^{1/2}}{\lambda X}.$$

It defines a holomorphic function on the open disk of center 0 and radius $(c_0\lambda)^{-1}$, which is bounded on the closure of that disk, therefore $c_n \leq \text{const}\,(c_0\lambda)^n$ for all $n \geq 0$. From that we deduce the uniform convergence of the series of analytic functions $\sum \hat{v}_n$ in $\mathcal{D}_\rho$ and an exponential bound for the sum:

$$\forall \zeta \in \mathcal{D}_\rho, \quad |\hat{v}(\zeta)| \leq \text{const } \mathrm{e}^{(c_0\lambda)^{1/2}|\zeta|}.$$

For $\hat{u}(\zeta) = -6\zeta + \hat{v}(\zeta)$ we can choose $\tau = (c_0\lambda)^{1/2}$ and write $|\hat{u}(\zeta)| \leq \text{const}\,|\zeta|\,\mathrm{e}^{\tau|\zeta|}$ in $\mathcal{D}_\rho$ (since $\hat{u}(\zeta) = \mathcal{O}(\zeta)$). But we can improve this bound by considering the equation we started with:

$$\forall \zeta \in \mathcal{D}_\rho \setminus \{0\}, \quad \hat{u}(\zeta) = -\beta(\zeta)(\hat{u} * \hat{u})(\zeta).$$

We know indeed that, if $|\zeta| > \rho$, $|\beta(\zeta)| \leq \text{const}\,\mathrm{e}^{-|\mathrm{Re}\,\zeta|}$, and $|\mathrm{Re}\,\zeta| \geq 2\delta|\zeta|$ in $\mathcal{D}_\rho$ with $\delta = (\rho^2 + 4\pi^2)^{-1/2}\rho/2$. Let us introduce a number $C > 0$ such that

$$\forall \zeta \in \mathcal{D}_\rho, \quad |\zeta|^2 |\beta(\zeta)| \leq C\,\mathrm{e}^{-\delta|\zeta|}.$$

We now see that any exponential bound

$$\forall \zeta \in \mathcal{D}_\rho, \quad |\hat{u}(\zeta)| \le C_0 |\zeta| \, \mathrm{e}^{\tau|\zeta|},$$

with $C_0 > 0$ and $\tau \in \mathbb{R}$, implies $|(\hat{u} * \hat{u})(\zeta)| \le \frac{C_0}{3!} |\zeta|^3 \, \mathrm{e}^{\tau|\zeta|}$, and thus

$$\forall \zeta \in \mathcal{D}_\rho, \quad |\hat{u}(\zeta)| \le \frac{C_0 C}{3!} |\zeta| \, \mathrm{e}^{(\tau-\delta)|\zeta|}.$$

This allows to decrease the exponential type τ indefinitely, and we conclude that *for all* $\tau \in \mathbb{R}$, the function $|\zeta|^{-1} \, \mathrm{e}^{-\tau|\zeta|} |\hat{u}(\zeta)|$ is bounded in $\mathcal{D}_\rho$. □

4.3 Convergence on the nearby sheets.

We now explore farther the Riemann surface $\mathcal{R}$, but still progressively. With respect to Section 4.2, some more geometrical facts are involved, but the analysis is quite similar.

Let $M \in \mathbb{N}^*$ and $\rho \in]0, \frac{2\pi}{2M+1}[$. We define the disks $D_1, \dots, D_{M+1}$ and the opposite disks $D_{-1}, \dots, D_{-M-1}$ by

$$D_m = D(2\pi \mathrm{i} m, m\rho), \quad D_{-m} = D(-2\pi \mathrm{i} m, m\rho), \quad m = 1, \dots, M.$$

We define $\mathcal{D}_{\rho,M}$ to be the closed set obtained by removing from $\mathbb{C}$ all these disks:

$$\mathcal{D}_{\rho,M} = \mathbb{C} \setminus \Big(\bigcup_{-M \le m \le M, \, m \ne 0} D_m \Big).$$

We define $\mathcal{R}^{(1)}_{\rho,M}$ to be the subset of $\mathcal{R}^{(1)}$ consisting of all the points ζ which can be represented by a path contained in $\mathcal{D}_{\rho,M}$ and such that the shortest such path γ_ζ is either

1. a straight segment;
2. or the union of a straight segment issuing from the origin and tangent to some disk D_m ($-M \le m \le M$, $m \ne 0$) and of an arc of the circle ∂D_m ending at $\overset{\bullet}{\zeta}$, and we require in that situation that the half-line $L(\zeta)$ tangent to γ_ζ at $\overset{\bullet}{\zeta}$ and going backwards be contained in $\mathcal{D}_{\rho,M}$;
3. or the union of a straight segment issuing from the origin and tangent to some disk D_m ($-M \le m \le M$, $m \ne 0$), of an arc of the circle ∂D_m, and of a straight segment $S(\zeta)$ tangent to D_m, ending at $\overset{\bullet}{\zeta}$ and such that the half-line $L(\zeta)$ which extends $S(\zeta)$ backwards from $\overset{\bullet}{\zeta}$ be contained in $\mathcal{D}_{\rho,M}$.

In the first case ζ lies in the main sheet $\mathcal{R}^{(0)}$, but in the last case it lies in the half-sheet contiguous to $\mathcal{R}^{(0)}$ corresponding to one crossing of $]2\pi \mathrm{i} m, 2\pi \mathrm{i}(m+1)[$ if $m \ge 1$ (of $]2\pi \mathrm{i}(m-1), 2\pi \mathrm{i} m[$ if $m \le -1$). In fact only a sector of this half-sheet is accessible because of the restriction $L(\zeta) \subset \mathcal{D}_{\rho,M}$ (see Fig. 4).

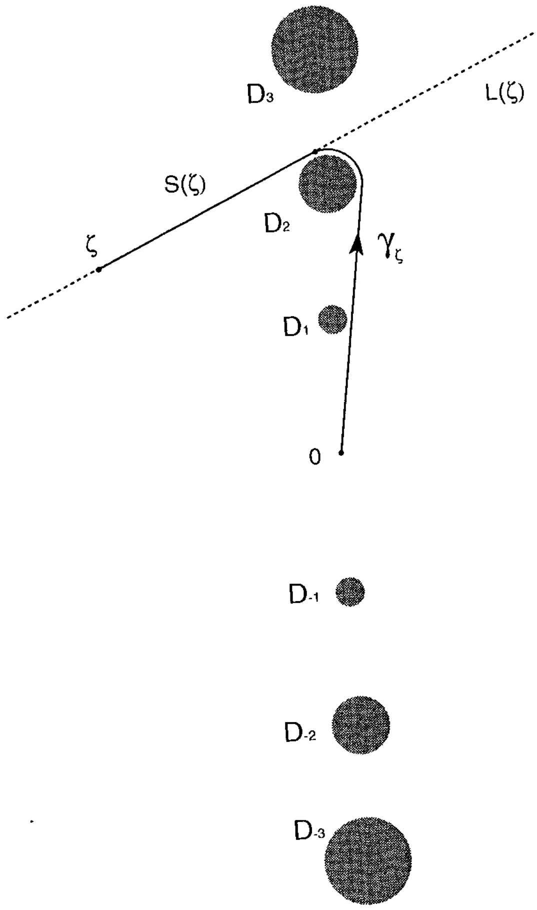

Figure 4: The paths γ_ζ

We have by construction

$$\mathcal{R}^{(1)} = \bigcup_{\rho\in]0,\pi/2[,\ M\in\mathbb{N}^*} \mathcal{R}^{(1)}_{\rho,M}.$$

We now fix for the rest of this section $M \in \mathbb{N}^*$ and $\rho \in]0, \frac{2\pi}{2M+1}[$. Our goal is to prove the uniform convergence of the series $\sum \hat{v}_n$ in $\mathcal{R}^{(1)}_{\rho,M}$.

We need to recall how one follows the analytic continuation of the convolution product of two holomorphic functions of $\mathcal{R}$, and to exhibit bounds which generalize Part (a) of Lemma 7. To that end we define, for each $\zeta \in \mathcal{R}^{(1)}_{\rho,M}$, a particular path Γ_ζ which is homotopic to γ_ζ and represents thus the same point ζ.

The path Γ_ζ is obtained by a deformation of γ_ζ which makes it symmetrically contractile. One can visualize its construction by letting a point ζ_1 move along γ_ζ from the origin to ζ, the point ζ_1 remaining connected to the origin by an extensible thread, and imagining fixed nails pointing upwards at the points of $2\pi i\mathbb{Z}$, with diameter $2|m|\rho$ for the nail at $2mi\pi$, and moving nails pointing downwards at the points of $\zeta_1 + 2\pi i\mathbb{Z}$ (with diameter $2|m|\rho$ for the nail at $\zeta_1 - 2mi\pi$) between which the thread is stretched progressively when ζ_1 moves along γ_ζ: at the end of the process ζ_1 has reached ζ and Γ_ζ is the thread under its final form. (One can think that the fixed nails remain on a fixed rule, and the moving nails are fastened to another rule which is parallel to the first one with reverse orientation and which is trailed by ζ_1 in its motion.) Notice that at each moment of the process the thread between the origin and ζ_1 remains symmetric with respect to its midpoint, thus Γ_ζ is symmetric and symmetrically contractile.

The previous construction applies to paths which are more general than the paths γ_ζ and which lead to points lying in $\mathcal{R}$ but not necessarily in $\mathcal{R}^{(1)}_{\rho,M}$. In our case, for a given point $\zeta \in \mathcal{R}^{(1)}_{\rho,M}$, the resulting path Γ_ζ is easily described according to the three possible shapes of γ_ζ (see Figure 5):

- in case 1 above, Γ_ζ coincides with γ_ζ;
- in case 3, if $m \geq 1$, the path Γ_ζ starts from the origin by a straight segment, meanders between the disks $\zeta - D_m$, $D_1, \ldots, \zeta - D_{m-k}$, $D_{k+1}, \ldots, \zeta - D_1$, D_m (in that order) and ends by a straight segment leading to ζ; moreover it is the shortest such path (if $m \leq -1$, D_{m-k} must be replaced by D_{m+k} $(1 \leq k \leq m-1)$ in the previous sentence); it is thus a succession of straight segments and arcs of circle;
- in case 2, the description is the same as in the previous case except that there is no straight segment from D_k to $\zeta - D_{m-k}$ for $k = 0, \ldots, m$ because of tangencies (with the convention $D_0 = \{0\}$).

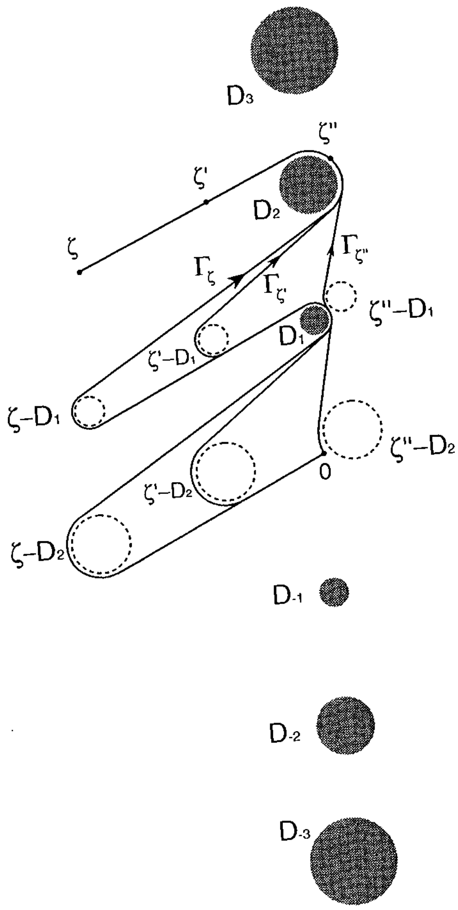

Figure 5: The paths Γ_ζ

The paths γ_ζ and Γ_ζ can be viewed as subsets of $\mathcal{R}^{(1)}_{\rho,M}$ rather than subsets of $\mathcal{D}_{\rho,M}$ (*i.e.* we identify them with their lifts in $\mathcal{R}$). Since Γ_ζ is symmetrically contractile, one can follow the analytic continuation at ζ of the convolution product $\hat{F} * \hat{G}$ of two germs $\hat{F}, \hat{G}$ which extend analytically to $\mathcal{R}^{(1)}_{\rho,M}$, and write it as

$$(\hat{F} * \hat{G})(\zeta) = \int_{\Gamma_\zeta} \hat{F}(\zeta_1)\hat{G}(\zeta_2)\, \mathrm{d}\zeta_1,$$

where ζ_2 is determined as the symmetric point of ζ_1 on Γ_ζ. Let us denote by s_ζ the curvilinear abscissa on Γ_ζ, by M_ζ the corresponding parametrization of Γ_ζ and by $\ell(\zeta)$ the length of Γ_ζ: we have $\ell(\zeta) = s_\zeta(\zeta)$ and the maps

$$\left\{ \begin{array}{ccc} \Gamma_\zeta & \longrightarrow & [0, \ell(\zeta)] \\ \zeta_1 & \longmapsto & s_\zeta(\zeta_1) \end{array} \right. \quad \text{and} \quad \left\{ \begin{array}{ccc} [0, \ell(\zeta)] & \longrightarrow & \Gamma_\zeta \\ s & \longmapsto & M_\zeta(s) \end{array} \right.$$

are mutually reciprocal. The formula for the analytic continuation of the convolution product may be written

$$(\hat{F} * \hat{G})(\zeta) = \int_0^{\ell(\zeta)} \hat{F}(M_\zeta(s))\hat{G}(M_\zeta(\ell(\zeta) - s)\Big(\frac{dM_\zeta}{ds}\Big)\, \mathrm{d}s.$$

Lemma 9 (New bounds for the convolution) *If $\hat{F}$ and $\hat{G}$ are holomorphic functions in $\mathcal{R}^{(1)}_{\rho,M}$ which satisfy*

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad |\hat{F}(\zeta)| \le \mathcal{F}(\ell(\zeta)) \quad \textit{and} \quad |\hat{G}(\zeta)| \le \mathcal{G}(\ell(\zeta)),$$

where $\mathcal{F}$ and $\mathcal{G}$ are continuous increasing functions on $\mathbb{R}^+$, their product of convolution $\hat{F} * \hat{G}$, which is holomorphic in $\mathcal{R}^{(1)}_{\rho,M}$, satisfies

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad |(\hat{F} * \hat{G})(\zeta)| \le (\mathcal{F} * \mathcal{G})(\ell(\zeta)).$$

Proof. The description of Γ_ζ given above allows one to check that

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \Gamma_\zeta \subset \mathcal{R}^{(1)}_{\rho,M}$$

and

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \; \forall \zeta_1 \in \Gamma_\zeta, \quad \ell(\zeta_1) \le s_\zeta(\zeta_1). \tag{20}$$

The conclusion then comes easily: since $\mathcal{F}$ and $\mathcal{G}$ are increasing, for $\zeta \in \mathcal{R}^{(1)}_{\rho,M}$ we have

$$\begin{aligned} |(\hat{F} * \hat{G})(\zeta)| &\le \int_0^{\ell(\zeta)} \mathcal{F}(\ell(M_\zeta(s)))\mathcal{G}(\ell(M_\zeta(\ell(\zeta) - s)))\, \mathrm{d}s \\ &\le \int_0^{\ell(\zeta)} \mathcal{F}(s_\zeta(M_\zeta(s)))\mathcal{G}(s_\zeta(M_\zeta(\ell(\zeta) - s)))\, \mathrm{d}s = (\mathcal{F} * \mathcal{G})(\ell(\zeta)). \end{aligned}$$

$\square$

Note that in this approach the inequality (20) is essential, but we know how to check such an inequality only for points in $\mathcal{R}^{(1)}_{\rho,M}$. One may then wonder whether it is possible to explore farther the Riemann surface $\mathcal{R}$ by a similar method or whether we are confined to the nearby sheets; we show in the appendix how to bypass the difficulty in order to explore every sheet of $\mathcal{R}$.

Lemma 10 (New bounds for the operator E**)** *There exist* $c, c_0 > 0$ *such that:*

(a)

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \left\{ \begin{array}{ll} |\beta(\zeta)| & \leq c^2 \ell(\zeta)^{-2}, \\ |Y(\zeta)| & \leq c\,\ell(\zeta)^{-5}, \\ |Z(\zeta)| & \leq c\,\ell(\zeta)^{2}, \end{array} \right.$$

and

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \hat{v}_0(\zeta) \leq c_0 \frac{\ell(\zeta)^3}{3!};$$

(b) if $\hat{W}$ *is holomorphic in* $\mathcal{R}^{(1)}_{\rho,M}$ *and satisfies*

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad |\hat{W}(\zeta)| \leq C \ell(\zeta)^{\nu}$$

for some real $C > 0$ *and integer* $\nu \geq 5$*, the function* $E.\hat{W}$ *is holomorphic in* $\mathcal{R}^{(1)}_{\rho,M}$ *and satisfies*

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad |(E.\hat{W})(\zeta)| \leq 2c^2 C \ell(\zeta)^{\nu-2}.$$

Proof. One checks the existence of a number $\kappa > 0$ such that

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \ell(\zeta) \leq \kappa(|\dot{\zeta}| + 1).$$

On the other hand, for $\zeta \in \mathcal{R}^{(1)}_{\rho,M}$ we have $|\dot{\zeta}| \leq \ell(\zeta)$ and

$$|\dot{\zeta}| \leq \rho \;\Rightarrow\; \zeta \in \mathcal{R}^{(0)} \;\Rightarrow\; \ell(\zeta) = |\dot{\zeta}|.$$

Thus Part (a) of Lemma 10 follows from Lemma 6.

Let $\hat{W}$, C, ν as in Part (b). The formula for the analytic continuation of $E.\hat{W}$ at a point ζ of $\mathcal{R}^{(1)}_{\rho,M}$ may be written

$$\begin{aligned} (E.\hat{W})(\zeta) \;=\; & \beta(\zeta)\hat{W}(\zeta) \\ & + \frac{1}{12} Z(\zeta) \int_0^{\ell(\zeta)} (Y\hat{W})(M_\zeta(s)) \Big(\frac{dM_\zeta}{ds}\Big)\, ds \\ & - \frac{1}{12} Y(\zeta) \int_0^{\ell(\zeta)} (Z\hat{W})(M_\zeta(s)) \Big(\frac{dM_\zeta}{ds}\Big)\, ds. \end{aligned}$$

Let us treat separately these three terms, using the inequalities of Part (a):
- the first term is bounded by $c^2 C\ell(\zeta)^{\nu-2}$;
- we observe that

$$|(Y\hat{W})(M_\zeta(s))| \leq cC\ell(M_\zeta(s))^{\nu-5} \leq cCs^{\nu-5}$$

because of the inequality (20) and the hypothesis $\nu \geq 5$, thus the second term is bounded by $\frac{c^2 C}{12(\nu-4)}\ell(\zeta)^{\nu-2}$;
- analogously

$$|(Z\hat{W})(M_\zeta(s))| \leq cC\ell(M_\zeta(s))^{\nu+2} \leq cCs^{\nu+2}$$

thus the third term is bounded by $\frac{c^2 C}{12(\nu+3)}\ell(\zeta)^{\nu-2}$.
Hence the desired bound for $|(E.\hat{W})(\zeta)|$. □

Lemma 11 (Convergence in the nearby sheets) *Let $c, c_0 > 0$ as in the previous lemma. The formulas*

$$c'_n = \sum_{n_1+n_2=n-1} c_{n_1} c_{n_2}, \quad c_n = \frac{c^2}{21} c'_n, \qquad n \geq 1$$

define inductively two sequences of positive numbers satisfying

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad |\hat{v}_n(\zeta)| \leq c_n \frac{\ell(\zeta)^{2n+3}}{(2n+3)!} \quad \text{and} \quad |\hat{w}_n(\zeta)| \leq c'_n \frac{\ell(\zeta)^{2n+5}}{(2n+5)!}.$$

The series of functions $\sum \hat{v}_n$ converges uniformly in $\mathcal{R}^{(1)}_{\rho,M}$ to a holomorphic function $\hat{v}$ and $\hat{u} = -6\zeta + \hat{v}$ has exponential decay at infinity in $\mathcal{R}^{(1)}_{\rho,M}$.

Proof. The desired inequalities are obtained exactly in the same way as those of Lemma 8. This proves the convergence of the series of functions $\sum \hat{v}_n$, and $\hat{u}$ is thus holomorphic in $\mathcal{R}^{(1)}_{\rho,M}$ with an exponential bound

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad |\hat{u}(\zeta)| \leq \text{const}\, \ell(\zeta)\, e^{\tau \ell(\zeta)},$$

where $\tau = (4c_0 c^2/21)^{1/2}$. As in the end of the proof of Lemma 8, we can improve this bound and decrease the exponential type τ, but this time the implication

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \ |\hat{u}(\zeta)| \leq C_0\, \ell(\zeta)\, e^{\tau \ell(\zeta)} \Rightarrow \forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \ |(\hat{u} * \hat{u})(\zeta)| \leq \frac{C_0}{3!} \ell(\zeta)^3\, e^{\tau \ell(\zeta)}$$

is ensured by Lemma 9 only for $\tau \geq 0$; introducing numbers $\delta, C > 0$ such that

$$\forall \zeta \in \mathcal{R}^{(1)}_{\rho,M}, \quad \ell(\zeta)^2 |\beta(\zeta)| \leq C\, e^{-\delta \ell(\zeta)},$$

we thus can reach $|\hat{u}(\zeta)| \leq \text{const}\, \ell(\zeta)\, e^{(\tau-\delta)\ell(\zeta)}$ with $\tau > 0$ and $\tau - \delta < 0$, but we must then stop. □

In fact, it is a consequence of the resurgent roperties of $\hat{u}$ explained in the appendix that it has exponential type $-\infty$ in $\mathcal{R}^{(1)}_{\rho,M}$ too.

5 Proof of Theorem 3

Let us use the same notations as in the previous section: $\hat{u}(\zeta) = -6\zeta + \hat{v}(\zeta) = -6\zeta + \sum \hat{v}_n(\zeta)$. We will obtain by induction the shape of the singularity at $2\pi\mathrm{i}$ for each $\hat{v}_n$, and the property of convergence established in the previous section will yield the result.

Definition 5.1

- *We say that a germ $\hat{F} \in \mathbb{C}\{\zeta\}$ is of type (-1) if it is odd and of valuation 5 at least, and if it extends analytically to $\mathcal{R}^{(1)}$ and can be written*

$$\hat{F}(\zeta) = \frac{B}{\zeta - 2\pi\mathrm{i}} + \frac{\hat{H}(\zeta - 2\pi\mathrm{i})}{2\pi\mathrm{i}} \log(\zeta - 2\pi\mathrm{i}) + \hat{R}(\zeta - 2\pi\mathrm{i})$$

  in a neighborhood of $2\pi\mathrm{i}$ on the main sheet, where $B \in \mathbb{C}$, and $\hat{H}$ and $\hat{R}$ are holomorphic at the origin with $\hat{H}(\xi) = C\xi + D\xi^3 + \mathcal{O}(\xi^5)$ for some $C, D \in \mathbb{C}$.

- *We say that a germ $\hat{F} \in \mathbb{C}\{\zeta\}$ is of type (-5) if it is odd and of valuation 3 at least, and if it extends analytically to $\mathcal{R}^{(1)}$ and can be written*

$$\begin{aligned}\hat{F}(\zeta) \;=\; & \frac{B}{(\zeta - 2\pi\mathrm{i})^5} + \frac{C}{(\zeta - 2\pi\mathrm{i})^3} + \frac{D}{\zeta - 2\pi\mathrm{i}} \\ & + \frac{\hat{H}(\zeta - 2\pi\mathrm{i})}{2\pi\mathrm{i}} \log(\zeta - 2\pi\mathrm{i}) + \hat{R}(\zeta - 2\pi\mathrm{i})\end{aligned}$$

  in a neighborhood of $2\pi\mathrm{i}$ on the main sheet, where $B, C, D \in \mathbb{C}$, and $\hat{H}$ and $\hat{R}$ are holomorphic at the origin with $\hat{H}(\xi) = \mathcal{O}(\xi)$.

Remark. One can rephrase the above definition using the *alien derivation* $\Delta_{2\pi i}$ of Resurgence theory (see the appendix): an odd germ $\hat{F} \in \mathbb{C}\{\zeta\}$ corresponds to an even formal series $\tilde{F}(z) \in \mathbb{C}[[z^{-1}]]$ via formal Borel-Laplace transform, and the requirements on the shape of the singularity at $2\pi\mathrm{i}$ amount respectively to the conditions

$$\Delta_{2\pi\mathrm{i}} \tilde{F} = 2\pi\mathrm{i} B + C z^{-2} + D z^{-4} + \mathcal{O}(z^{-6})$$

and

$$\Delta_{2\pi\mathrm{i}} \tilde{F} = \frac{2\pi\mathrm{i}}{4!} B z^4 + \frac{2\pi\mathrm{i}}{2!} C z^2 + \frac{2\pi\mathrm{i}}{0!} D + \mathcal{O}(z^2).$$

Lemma 12 (Transformation of singularities) *The convolution product of two germs of type (-5) is of type (-1), and the image by the operator E of a germ of type (-1) is of type (-5).*

Proof. Let us consider two germs $\hat{F}_1$ and $\hat{F}_2$ of type (-5): their convolution product $\hat{G}$ is odd and of valuation 7 at least, and it extends analytically to $\mathcal{R}^{(1)}$. One checks that its singularity at $2\pi\mathrm{i}$ has the desired form by a direct analysis of the convolution integral, writing it as

$$\hat{G}(\zeta) = \int_{\zeta/2}^{\zeta} \hat{F}_1(\zeta_1)\hat{F}_2(\zeta-\zeta_1)\,\mathrm{d}\zeta_1 + \int_{\zeta/2}^{\zeta} \hat{F}_1(\zeta-\zeta_2)\hat{F}_2(\zeta_2)\,\mathrm{d}\zeta_2$$

like in the proof of Theorem 4.

Alternatively one can use the framework of Resurgence and the fact that the operator $\Delta_{2\pi i}$ satisfies the Leibniz rule: the formal series $\tilde{G}(z)$ associated with $\hat{G}(\zeta)$ is the product of the formal series $\tilde{F}_1$ and $\tilde{F}_2$ associated with our germs,

$$\Delta_{2\pi\mathrm{i}}\tilde{G} = \tilde{F}_1\Delta_{2\pi\mathrm{i}}\tilde{F}_2 + \tilde{F}_2\Delta_{2\pi\mathrm{i}}\tilde{F}_1,$$

and for $j=1,2$ we have $\tilde{F}_j(z) = \mathcal{O}(z^{-4})$, even, whereas $\Delta_{2\pi\mathrm{i}}\tilde{F}_j(z) = B_j z^4 + C_j z^2 + D_j + \mathcal{O}(z^{-2})$ for some complex numbers B_j, C_j, D_j, hence the result.

Let us now consider a germ $\hat{F}$ of type (-1). We have already noticed that $\hat{G} = E.\hat{F}$ is of valuation 3 at least and extends analytically to $\mathcal{R}^{(1)}$; its is easily seen to be odd. Let us study its singularity at $2\pi\mathrm{i}$. We use the expression

$$\hat{G} = Y\int_0 \Big(y^{-2}\int_0 y\hat{F}''\Big) \quad \text{with} \quad y = \alpha Y/12,$$

which can be checked from the proof of Lemma 4. For ξ small and such that $2\pi\mathrm{i}+\xi$ lies in the main sheet $\mathcal{R}^{(0)}$, we can write

$$\hat{F}''(2\pi\mathrm{i}+\xi) = *\,\xi^{-3} + *\,\xi^{-1} + \xi(* + \mathcal{O}(\xi^2))\log\xi + \mathrm{reg}(\xi),$$

where the stars $*$ stand for some complex numbers and $\mathrm{reg}(\xi)$ denotes some regular germ. But $y(2\pi\mathrm{i}+\xi) = *\,\xi^{-3}(1+\mathcal{O}(\xi^2))$ is odd, thus

$$\begin{aligned}(y\hat{F}'')(2\pi\mathrm{i}+\xi) &= *\,\xi^{-6} + *\,\xi^{-4} + *\,\xi^{-3} + *\,\xi^{-2} + *\,\xi^{-1}\\ &\quad +(*\,\xi^{-2} + \mathrm{reg}(\xi))\log\xi + \mathrm{reg}(\xi)\end{aligned}$$

and

$$\Big(\int_0 y\hat{F}''\Big)(2\pi\mathrm{i}+\xi) = \xi^{-5}(* + \mathcal{O}(\xi^2)) + \xi^{-1}\mathrm{reg}(\xi)\log\xi.$$

Now $y^{-2}(2\pi\mathrm{i}+\xi) = *\,\xi^6(1+\mathcal{O}(\xi^2))$ is even, thus $(y^{-2}\int_0 y\hat{F}'')(2\pi\mathrm{i}+\xi) = \xi(*+\mathcal{O}(\xi^2)) + \xi^5\mathrm{reg}(\xi)\log\xi$,

$$\Big[\int_0\Big(y^{-2}\int_0 y\hat{F}''\Big)\Big](2\pi\mathrm{i}+\xi) = * + *\,\xi^2 + \mathcal{O}(\xi^4) + \xi^6\mathrm{reg}(\xi)\log\xi,$$

and since $Y(2\pi\mathrm{i}+\xi) = *\,\xi^{-5}(1+\mathcal{O}(\xi^2))$ is even, we conclude that

$$\hat{G}(2\pi\mathrm{i}+\xi) = *\,\xi^{-5} + *\,\xi^{-3} + *\,\xi^{-1} + \xi\mathrm{reg}(\xi)\log\xi + \mathrm{reg}(\xi)$$

□

as required.

Since $\hat{w}_0$ extends to an entire function, it follows easily by induction that each $\hat{v}_n$ is of type (-5) and that each $\hat{w}_n$ is of type (-1). Thus there exist sequences of numbers $(A_5^{(n)})$, $(A_3^{(n)})$ $(A_1^{(n)})$ and sequences of functions $(h^{(n)})$, $(r^{(n)})$ holomorphic near the origin such that, for all $n \geq 0$,

$$\hat{v}_n(2\pi\mathrm{i} + \xi) = \frac{A_5^{(n)}}{\xi^5} + \frac{A_3^{(n)}}{\xi^3} + \frac{A_1^{(n)}}{\xi} + \frac{1}{2\pi\mathrm{i}} h^{(n)}(\xi) \log \xi + r^{(n)}(\xi),$$

for $\zeta = 2\pi\mathrm{i} + \xi$ close to $2\pi\mathrm{i}$ on the main sheet, with $h^{(n)}(0) = 0$.

For any $n \geq 0$, the function $h^{(n)}$ is nothing but the "variation" (or monodromy) of the singularity of $\hat{v}_n$ around $2\pi\mathrm{i}$:

$$h^{(n)}(\xi) = \hat{v}_n(2\pi\mathrm{i} + \xi) - \hat{v}_n(2\pi\mathrm{i} + \xi.\mathrm{e}^{-2\pi\mathrm{i}})$$

for $\zeta = 2\pi\mathrm{i} + \xi$ close to $2\pi\mathrm{i}$ on the main sheet, if we denote by $2\pi\mathrm{i} + \xi.\mathrm{e}^{-2\pi\mathrm{i}}$ the point of $\mathcal{R}$ with the same projection onto $\mathbb{C}$ but lying in the sheet immediately "below" the main one (*i.e.* ζ is represented by the segment $[0, \dot{\zeta}]$, but $2\pi\mathrm{i}+\xi.\mathrm{e}^{-2\pi\mathrm{i}}$ is represented by the path which begins by the straight segment and continues by a clockwise-oriented circle around $2\pi\mathrm{i}$). But then Lemma 11 implies the uniform convergence of the series $\sum h^{(n)}$ in a disk $D(O, \rho_0)$ centered at the origin and of sufficiently small radius ρ_0:

$$h = \sum_{n \geq 0} h^{(n)}$$

is holomorphic at the origin and satisfies $h(0) = 0$.

Now consider the functions

$$\hat{v}_n^*(\xi) = \frac{A_5^{(n)}}{\xi^5} + \frac{A_3^{(n)}}{\xi^3} + \frac{A_1^{(n)}}{\xi} + r^{(n)}(\xi)$$

for $n \geq 0$: they are holomorphic in the pointed disk $\{\, 0 < |\xi| < \rho_0 \,\}$ and the series $\sum \hat{v}_n^*$ is uniformly convergent in the annulus $D(0, \rho_0) \setminus D(0, \rho)$ for all $\rho \in]0, \rho_0[$; its sum

$$\hat{v}^*(\xi) = \sum_{n \geq 0} \hat{v}_n^*(\xi) = \hat{v}(2\pi\mathrm{i} + \xi) - \frac{1}{2\pi\mathrm{i}} h(\xi) \log \xi$$

is holomorphic in the pointed disk $D(0, \rho_0) \setminus \{0\}$. Writing the coefficients $A_5^{(n)}, A_3^{(n)}, A_1^{(n)}$ as Cauchy integrals involving $\hat{v}_n^*$, we thus deduce that the series

$$A_5 = \sum_{n \geq 0} A_5^{(n)}, \quad A_3 = \sum_{n \geq 0} A_3^{(n)}, \quad A_1 = \sum_{n \geq 0} A_1^{(n)}$$

are convergent and conclude the proof of Theorem 3 by observing that the function $\xi \mapsto \hat{v}^*(\xi) - A_5 \xi^{-5} - A_3 \xi^{-3} - A_1 \xi^{-1}$ is regular at the origin.

6 Proof of Theorem 4

Let us dwell on the derivation of the equation for φ. We consider the function $\hat{u}$ near its singularity $2\pi\mathrm{i}$ at the first sheet of the Riemann surface $\mathcal{R}$. Consider a point ζ on the main sheet of the Riemann surface $\mathcal{R}$ close to the singularity at $2\pi\mathrm{i}$. We note that

$$(\hat{u} * \hat{u})(\zeta) = 2\int_{\zeta/2}^{\zeta} \hat{u}(\zeta-\zeta')\hat{u}(\zeta')\,\mathrm{d}\zeta',$$

where the integral is taken over a rectilinear segment. In this way we separate the singular and the regular factors: when ζ is close to $2\pi\mathrm{i}$ the argument of the first function, $\hat{u}(\zeta-\zeta')$, remains far from the singularity. The convolution equation (6) takes the form,

$$4\sinh^2\frac{\zeta}{2}\,u(\zeta) = -2\int_{\zeta/2}^{\zeta} \hat{u}(\zeta-\zeta')\hat{u}(\zeta')\,\mathrm{d}\zeta'. \tag{21}$$

Now we use the first two terms of expansions (5) and (7) to evaluate $\hat{u}(\zeta-\zeta')$ and $\hat{u}(\zeta')$, respectively. We have

$$\begin{aligned}
-2\int_{\zeta/2}^{\zeta} \hat{u}(\zeta-\zeta')\hat{u}(\zeta')\,\mathrm{d}\zeta' &= -\frac{a_1A_5}{6}\frac{1}{\xi^3} - \left(a_1A_3 + \frac{a_2A_5}{12}\right)\frac{1}{\xi} + \mathcal{O}(1)\\
&= \frac{A_5}{\xi^3} + \left(6A_3 - \frac{15A_5}{24}\right)\frac{1}{\xi} + \mathcal{O}(1),
\end{aligned}$$

where $\xi = \zeta - 2\pi\mathrm{i}$. On the other hand,

$$4\sinh^2\frac{\zeta}{2}\,u(\zeta) = \frac{A_5}{\xi^3} + \left(A_3 + \frac{A_5}{12}\right)\frac{1}{\xi} + \mathcal{O}(\xi).$$

Comparing the last two equations we conclude that (21) implies

$$A_3 = \frac{17A_5}{120}.$$

We obtain the third polar coefficients A_1 and the function h from the analysis of the variation (monodromy) of $\hat{u}$.

Let two points, denoted by ζ_1 and ζ_2, converge to the imaginary axis just above the singularity at $2\pi\mathrm{i}$ from the right-hand-side and from the left-hand-side, respectively. Let $\zeta = 2\pi\mathrm{i} + \xi$ denote the limit point (see Fig. 6). Then the prelogarithmic factor of (7) is given by

$$h(\xi) = \lim_{\substack{\zeta_1 \to 2\pi\mathrm{i}+\xi+0\\ \zeta_2 \to 2\pi\mathrm{i}+\xi-0}} u(\zeta_1) - u(\zeta_2).$$

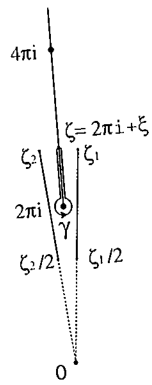

Figure 6: Integral paths

In order to evaluate the limit we take the difference of the two copies of the convolution equation,

$$4\sinh^2\frac{\zeta_k}{2}\,u(\zeta_k) = -2\int_{\zeta_k/2}^{\zeta_k}\hat{u}(\zeta_k-\zeta')\hat{u}(\zeta')\,\mathrm{d}\zeta'\,,$$

and pass to the limit:

$$\begin{aligned}4\sinh^2\frac{\xi}{2}\,h(\xi) &= -2\int_\gamma \hat{u}(\zeta-\zeta')\hat{u}(\zeta')\,\mathrm{d}\zeta' \\ &= -2\int_{\xi'+2\pi\mathrm{i}\in\gamma}\hat{u}(\xi-\xi')\hat{u}(2\pi\mathrm{i}+\xi')\,\mathrm{d}\xi'\,.\end{aligned}$$

Now we substitute the convergent expansion (7) instead of $\hat{u}(2\pi\mathrm{i}+\xi)$:

$$\begin{aligned}4\sinh^2\frac{\xi}{2}\,h(\xi) &= -2\int_{\xi'+2\pi\mathrm{i}\in\gamma}\hat{u}(\xi-\xi')\frac{\log\xi'}{2\pi\mathrm{i}}h(\xi')\,\mathrm{d}\xi' \\ &\quad -4\pi\mathrm{i}\operatorname*{Res}_{\xi'=0}\left[\hat{u}(\xi-\xi')\left(\frac{A_5}{\xi'^5}+\frac{A_3}{\xi'^3}+\frac{A_1}{\xi'}\right)\right] \\ &= -2\int_0^\xi \hat{u}(\xi-\xi')h(\xi')\,\mathrm{d}\xi' \\ &\quad -4\pi\mathrm{i}\left(A_5\frac{\partial_\xi^4\hat{u}(\xi)}{4!}+A_3\frac{\partial_\xi^2\hat{u}(\xi)}{2!}+A_1\hat{u}(\xi)\right)\,.\end{aligned}$$

In this way we obtain the following equation on the singularity of $\hat{u}$:

$$4\sinh^2\frac{\xi}{2}\,h(\xi) = -2(\hat{u}*h)(\xi) - f(\xi)\,, \tag{22}$$

where

$$f(\xi) = 4\pi \mathrm{i} \left(A_5 \frac{\partial_\xi^4 \hat{u}(\xi)}{4!} + A_3 \frac{\partial_\xi^2 \hat{u}(\xi)}{2!} + A_1 \hat{u}(\xi) \right) .$$

This is a linear nonhomogeneous equation for h. Since $h(0) = \hat{u}(0) = 0$ the corresponding terms of the equation are cubic at zero. Consequently $f(0) = f'(0) = f''(0) = 0$. The equality $f'(0) = 0$ implies

$$A_5 a_3/4! + A_3 a_2/2! + A_1 a_1 = 0 .$$

This implies $A_1 = -\frac{17}{640} A_5$.

To finish the proof it is sufficient to check that Eq. (22) generates exactly the same recurrence rule for h_k as the formal homogeneous variational equation (8), which is not too difficult. □

A Resurgence of formal solutions

A.1 A (more) general formal solution

We now give an overview of the resurgent structure of all the formal series which have been introduced in this article and we complete the proof of Theorem 2. We first extend the notion of formal solution for the equation (3).

Proposition 2 (Normalized general solution) *There is a unique sequence of nonzero series $(\tilde{u}_n)_{n\in\mathbb{N}}$ in $\mathbb{C}[z][[z^{-1}]]$ such that:*

- *the series*

$$\tilde{u}(z, b) = \sum_{n\geq 0} b^n \tilde{u}_n(z)$$

  satisfies formally (3) when expanding both sides of the equation in powers of b and then in powers of z;
- *each series $\tilde{u}_n$ is even;*
- *$\tilde{u}_1(z) = z^4 + \mathcal{O}(z^2)$;*
- *for all $n \geq 2$, the coefficient of z^4 in $\tilde{u}_n$ is zero.*

Remark. A more general formal solution is obtained by considering

$$\tilde{u}(z + a(z), b(z)) = \sum_{n\geq 0} b(z)^n \tilde{u}_n(z + a(z))$$

where $a(z)$ and $b(z)$ are 1-periodic objects, *e.g.* formal expansions in powers of $\mathrm{e}^{2\pi \mathrm{i} z}$ (or of $\mathrm{e}^{-2\pi \mathrm{i} z}$, but not both at the same time).

Proof. Let us introduce notations for difference operators:

$$S : f(z) \mapsto f(z+1) - f(z), \qquad P : f(z) \mapsto f(z+1) - 2f(z) + f(z-1).$$

When a formal Laurent series $f \in \mathbb{C}[z][[z^{-1}]]$ is given, it admits a primitive in $\mathbb{C}[z][[z^{-1}]]$ if and only if its residuum (the coefficient of z^{-1} in f) vanishes; in that case we denote by $\partial_z^{-1} f$ the unique primitive of f without constant term. The invertibility of S is easily studied:

Lemma 13 *A formal Laurent series $f \in \mathbb{C}[z][[z^{-1}]]$ admits a preimage by S in $\mathbb{C}[z][[z^{-1}]]$ if and only if its residuum vanishes. In that case the unique preimage of f without constant term can be obtained as*

$$S^{-1} f = B(\partial_z)\partial_z^{-1} f,$$

where

$$B(X) = \frac{X}{e^X - 1} = 1 - \frac{X}{2} + \sum_{\ell \geq 1} (-1)^{\ell+1} B_\ell \frac{X^{2\ell}}{(2\ell)!}.$$

(The proof is straightforward.)

When substituting $\tilde{u}(z,b)$ inside (3) and expanding with respect to b both sides of equation, we find

- $P\tilde{u}_0 = -\tilde{u}_0^2$,
- $P\tilde{u}_1 = -2\tilde{u}_0\tilde{u}_1$,
- $P\tilde{u}_n = -\sum_{n_1+n_2=n} \tilde{u}_{n_1}\tilde{u}_{n_2}, \quad n \geq 2.$

We already know that the first of these equations admits a unique nonzero even solution $\tilde{u}_0$, which is nothing but the series called u in the rest of the paper:

$$\tilde{u}_0(z) = \sum_{k\geq 1} a_k z^{-2k} = -6z^{-2} + \frac{15}{2}z^{-4} - \frac{663}{40}z^{-6} + \frac{43647}{800}z^{-8} + \mathcal{O}(z^{-10}).$$

The second equation coincides with the variational equation (8) whose fundamental system of solutions (φ_1, φ_2) was introduced in Lemma 2, according to which there is only one possibility for $\tilde{u}_1$:

$$\tilde{u}_1(z) = 84\varphi_2(z) = z^4 + \frac{17}{10}z^2 - \frac{51}{80} + \frac{36}{5}z^{-2} + \mathcal{O}(z^{-4}).$$

We recall that

$$\varphi_1 = \partial_z \tilde{u}_0 = \sum_{k\geq 1} b_k z^{-2k-1}, \tag{23}$$

whereas $\varphi_2 = \sum_{k\geq -2} d_k z^{-2k}$ could be found directly. But one could also use a method which is the finite-difference analogue of the classical method of variation of parameters for second-order ordinary differential equations (see *e.g.* [Gel99] for detailed explanations); this leads to

$$\varphi_2 = \varphi_1 S^{-1}\chi, \quad \chi(z) = \frac{1}{\varphi_1(z)\varphi_1(z+1)} \tag{24}$$

(it can be checked that χ has no residuum since φ_1 is odd).

The next equations can be considered as linear non-homogeneous finite-difference equations: for $n \geq 2$, the series $\tilde{u}_n$ is required to satisfy

$$(P + 2\tilde{u}_0).\tilde{u}_n = \tilde{v}_n \tag{25}$$

with a right-hand side

$$\tilde{v}_n = -\sum_{k=1}^{n-1} \tilde{u}_k \tilde{u}_{n-k}$$

determined by the previous terms $\tilde{u}_0, \ldots, \tilde{u}_{n-1}$.

Lemma 14 *If a Laurent series $\psi \in \mathbb{C}[z^2][[z^{-2}]]$ is given such that $\varphi_1\psi$ has no residuum, the linear non-homogeneous equation*

$$(P + 2\tilde{u}_0).\varphi = \psi$$

admits a unique solution φ in $\mathbb{C}[z^2][[z^{-2}]]$ whose coefficient of z^4 vanishes. This solution can be written

$$\varphi(z) = \frac{1}{2}\Big((\Phi(z) + \Phi(-z)\Big) + c\varphi_2(z),$$

where c is some complex number and

$$\Phi = -\varphi_1 S^{-1}(\varphi_2\psi) + \varphi_2 S^{-1}(\varphi_1\psi).$$

(*Proof of the lemma:* Since the "Wronskian" of (φ_1, φ_2) is equal to 1, one can check that $\Phi = \alpha\varphi_1 + \beta\varphi_2$ is solution of the non-homogeneous equation as soon as $S\alpha = -\varphi_2\psi$ and $S\beta = \varphi_1\psi$. By hypothesis the series $\varphi_1\psi$ has no residuum, and the same is true for $\varphi_2\psi$ because φ_2 and ψ are even; we can thus choose $\alpha = -S^{-1}(\varphi_2\psi)$ and $\beta = S^{-1}(\varphi_1\psi)$, and we obtain a first solution Φ. But since $\tilde{u}_0$ and ψ are even, $\Phi(-z)$ is also solution of the non-homogeneous equation, therefore the odd series $\Phi(z) - \Phi(-z)$ satisfies the homogeneous equation and can be written $c_1\varphi_1(z) + c_2\varphi_2(z)$ with $c_1, c_2 \in \mathbb{C}$. Now $c_2 = 0$ because of oddness and

$$\Phi(z) = \frac{1}{2}\Big(\Phi(z) + \Phi(-z)\Big) + \frac{c_1}{2}\varphi_1(z).$$

The unique even solution φ without coefficient in front of z^4 is obtained by removing $\frac{c_1}{2}\varphi_1(z)$ and adding the appropriate multiple of $\varphi_2(z)$.)

We now proceed by induction in order to solve the equations (25). Let us suppose that, for some $n \geq 2$, the series $\tilde{u}_0, \dots, \tilde{u}_{n-1}$ have been determined in $\mathbb{C}[z^2][[z^{-2}]]$. The series $\tilde{v}_n = -\sum_{k=1}^{n-1} \tilde{u}_k \tilde{u}_{n-k}$ belongs to that space too, and we only have to check that $\varphi_1 \tilde{v}_n = \tilde{v}_n \partial_z \tilde{u}_0$ has no residuum. This results from the identity[3]

$$\tilde{v}_n \partial_z \tilde{u}_0 = -\frac{1}{2}\partial_z \Big[\tilde{u}_0 \sum_{k=1}^{n-1} \tilde{u}_k \tilde{u}_{n-k}\Big] - \frac{1}{2}\sum_{k=1}^{n-1}\Big[(\partial_z \tilde{u}_k)(P\tilde{u}_{n-k}) + (\partial_z \tilde{u}_{n-k})(P\tilde{u}_k)\Big],$$

since the derivative of a Laurent series has no residuum and, for any two Laurent series f and g,

$$(\partial_z f)(Pg) + (Pf)(\partial_z g) = \sum_{m\geq 0} \frac{2}{(2m+2)!}\Big[(\partial_z f)(\partial_z^{2m+2} g) + (\partial_z^{2m+2} f)(\partial_z g)\Big]$$

has no residuum due to

$$F(\partial_z^{2m+1} G) + (\partial_z^{2m+1} F)G = \partial_z \Big[\sum_{\ell=0}^{2m} (-1)^\ell (\partial_z^\ell F)(\partial_z^{2m-\ell} G)\Big].$$

□

Remark. In fact $\tilde{u}(z,b) = \sum_{m\geq 0} z^{-2m-2} U_m(bz^6)$, with a family of formal series $U_m(X) \in \mathbb{Q}[[X]]$. For instance $U_0(X)$ is the generating series for the leading terms:

$$U_0(X) = \sum_{n\geq 0} c_n X^n \quad \text{with} \quad \forall n \geq 0, \quad \tilde{u}_n(z) = c_n z^{6n-2} + \mathcal{O}(z^{6n-4}).$$

Its coefficients can be computed inductively since it is the unique nonzero formal solution of the equation

$$(6b\partial_b - 2)(6b\partial_b - 3)U_0 = -U_0^2.$$

It can be checked that all the series U_m have positive radius of convergence.

A.2 Resurgent properties of the general formal solution

Now comes the essential result of this appendix, which contains in fact Theorems 5 and 3. The following theorem is formulated in the language of Resurgence theory, but we provide some explanations on its meaning after its statement.

[3] A compact way of deriving this identity consists in introducing the generating series $U = \sum_{n\geq 1} b^n \tilde{u}_n$ and $V = \sum_{n\geq 2} b^n \tilde{v}_n$, and checking that

$$\partial_z \Big(\int_0^{U(z,b)} [\tilde{u}_0^2(z) - (\tilde{u}_0(z) + X)^2]\,\mathrm{d}X\Big) = (P.U)\partial_z U + (\partial_z \tilde{u}_0)V.$$

Theorem 6 (Resurgence of the general solution) *For each integer $n \geq 0$, the formal series $\tilde{u}_n$ is a simply ramified resurgent function whose minor extends analytically to $\mathcal{R}$, with a growth of exponential type $-\infty$ along the non-vertical rays of each half-sheet of $\mathcal{R}$.*

There exist two families of formal series

$$A_\omega(b) = \sum_{n\geq 0} A_{\omega||n} b^n, \quad B_\omega(b) = \sum_{n\geq 0} B_{\omega||n} b^n, \qquad \omega \in 2\pi \mathrm{i}\mathbb{Z}^*,$$

such that the Bridge equation holds:

$$\forall \omega \in 2\pi\mathrm{i}\mathbb{Z}^*, \quad \Delta_\omega \tilde{u}(z,b) = \Big(A_\omega(b)\partial_b + B_\omega(b)\partial_z\Big)\tilde{u}(z,b). \tag{26}$$

This equation must be understood as a compact writing of the resurgence relations

$$\forall \omega \in 2\pi\mathrm{i}\mathbb{Z}^*, \forall n \geq 0, \quad \Delta_\omega \tilde{u}_n = \sum_{n_1+n_2=n} \Big[(n_2+1)A_{\omega||n_1}\tilde{u}_{n_2+1} + B_{\omega||n_1}\partial_z \tilde{u}_{n_2}\Big] \tag{27}$$

which allow one to compute all the alien derivatives of the resurgent functions $\tilde{u}_n$.

A.3 Explanation of resurgent terminology and remarks

a) Simply ramified resurgent functions

If a formal Laurent series $\tilde{\varphi}(z) \in \mathbb{C}[z][[z^{-1}]]$ is given, we can isolate the polynomial part $\psi(z)$ and compute the formal Borel transform of the remainder $\tilde{\varphi}_0(z)$ according to the usual rule $\mathcal{B} : z^{-n-1} \mapsto \zeta^n/n!$:

$$\tilde{\varphi} = \psi + \varphi_0,$$

$$\begin{cases} \psi \in \mathbb{C}[z], \\ \tilde{\varphi}_0 \in z^{-1}\mathbb{C}[[z^{-1}]], \quad \hat{\varphi} = \mathcal{B}\tilde{\varphi}_0 \in \mathbb{C}[[\zeta]]. \end{cases}$$

Let us suppose that $\hat{\varphi}$ is a convergent power-series which defines a germ of analytic function which extends analytically to $\mathcal{R}$. In that situation, $\tilde{\varphi}$ is said to be a *resurgent function* and $\hat{\varphi}$ is called its *minor*.

Thus the first assertion in Theorem 6 constitutes a generalization of Theorem 5.

If moreover, when following the analytic continuation of the minor $\hat{\varphi}$, the only encountered singularities are of the form {polar part}+{logarithmic singularity}, the resurgent function $\tilde{\varphi}$ is said to be *simply ramified* (we mean that, if $\hat{\varphi}_\Gamma$ denotes the determination of $\hat{\varphi}$ obtained by following some path Γ of analytic continuation which leads close to some point ω of $2\pi\mathrm{i}\mathbb{Z}$, we can write

$$\hat{\varphi}_\Gamma(\omega+\zeta) = \mathrm{pol}(\zeta^{-1}) + \frac{1}{2\pi i}\mathrm{var}(\zeta)\log\zeta + \mathrm{reg}(\zeta),$$

with $\mathrm{pol}(X) \in \mathbb{C}[X]$ and $\mathrm{var}(\zeta), \mathrm{reg}(\zeta) \in \mathbb{C}\{\zeta\}$).

Simply ramified resurgent functions whose minors extends analytically to $\mathcal{R}$ form a subalgebra RES of $\mathbb{C}[z][[z^{-1}]]$.

b) Alien derivations

Let $\omega \in 2\pi\mathrm{i}\mathbb{Z}^*$. The *alien derivation of index* ω is a particular linear operator Δ_ω of RES which satisfies the Leibniz rule:

$$\forall \tilde\varphi_1, \tilde\varphi_2 \in \mathrm{RES}, \quad \Delta_\omega(\tilde\varphi_1\tilde\varphi_2) = (\Delta_\omega\tilde\varphi_1)\tilde\varphi_2 + \tilde\varphi_1(\Delta_\omega\tilde\varphi_2).$$

For definiteness, let us first consider the case where $\omega = 2\pi i r$ with $r \geq 1$: if $\tilde\varphi$ is given in RES, we may consider the 2^{r-1} determinations of the minor $\hat\varphi$ in the segment $]2\pi\mathrm{i}(r-1), 2\pi i r[$ which are obtained by following its analytic continuation along the half-line $i\mathbb{R}^+$ and circumventing the intermediary singular points $2\pi\mathrm{i}, \dots, 2\pi\mathrm{i}(r-1)$ to the left or to the right; we denote them

$$\hat\varphi^{\varepsilon_1,\dots,\varepsilon_{r-1}},$$

each ε_ℓ being a plus sign or a minus sign indicating whether $2\pi i\ell$ was circumvented to the left or to the right. For $\zeta \in]-2\pi\mathrm{i}, 0[$, we set

$$\check\chi(\zeta) = \sum_{\varepsilon_1,\dots,\varepsilon_{r-1}} \frac{p(\varepsilon)!q(\varepsilon)!}{r!} \varphi^{\varepsilon_1,\dots,\varepsilon_{r-1}}(2\pi\mathrm{i}r + \zeta),$$

where the integers $p(\varepsilon)$ and $q(\varepsilon) = r - 1 - p(\varepsilon)$ denote the numbers of plus signs and of minus signs in the sequence $(\varepsilon_1, \dots, \varepsilon_{r-1})$. According to our hypothesis on the shape of the singularities of the minor $\hat\varphi$, the function $\check\chi$ must take the form

$$\check\chi(\zeta) = A_N\zeta^{-N} + \cdots + A_1\zeta^{-1} + \frac{1}{2\pi\mathrm{i}}\hat\chi(\zeta)\log\zeta + \mathrm{reg}(\zeta),$$

where $A_1, \dots, A_N$ are some complex numbers and $\hat\chi(\zeta), \mathrm{reg}(\zeta) \in \mathbb{C}\{\zeta\}$. We define

$$\Delta_\omega\tilde\varphi = 2\pi\mathrm{i}\left[\tfrac{(-1)^{N-1}}{(N-1)!}A_N z^{N-1} + \cdots + A_1\right] + \mathcal{B}^{-1}\hat\chi.$$

It can be checked that $\Delta_\omega\tilde\varphi$ is a well-defined element of RES; observe that its minor $\hat\chi$ can be computed in the segment $]0, 2\pi\mathrm{i}[$ according to the formula $\forall\zeta \in]0, 2\pi\mathrm{i}[$,

$$\hat\chi(\zeta) = \sum_{\varepsilon_1,\dots,\varepsilon_{r-1}} \frac{p(\varepsilon)!q(\varepsilon)!}{r!}\left[\varphi^{\varepsilon_1,\dots,\varepsilon_{r-1},+}(2\pi\mathrm{i}r + \zeta) - \varphi^{\varepsilon_1,\dots,\varepsilon_{r-1},-}(2\pi\mathrm{i}r + \zeta)\right].$$

If $\omega = -2\pi\mathrm{i}r$ with $r \geq 1$, the operator Δ_ω is defined in a similar fashion.

If a simply ramified resurgent function $\tilde\varphi$ has only real coefficients, *i.e.* $\tilde\varphi(z) \in \mathrm{RES} \cap \mathbb{R}[z][[z^{-1}]]$, one checks that

$$\Delta_{-2\pi\mathrm{i}r}\tilde\varphi(z) = -\overline{(\Delta_{2\pi\mathrm{i}r}\tilde\varphi)(\overline{z})}.$$

On the other hand, if $\tilde{\varphi}$ is even with respect to z,

$$\Delta_{-2\pi i r}\tilde{\varphi}(z) = (\Delta_{2\pi i r}\tilde{\varphi})(-z).$$

The fact that the operators Δ_ω are derivations is essential in Resurgence theory. They are called alien derivations by contrast with the natural derivation ∂_z. There is a relation

$$\Delta_\omega \circ \partial_z = (\partial_z - \omega) \circ \Delta_\omega,$$

but no relation between the Δ_ω's themselves: they generate a free Lie algebra.

These operators encode in fact the whole singular behavior of the minors. Given a sequence $\omega_1, \ldots, \omega_n$ in $2\pi i\mathbb{Z}^*$, the composed operator $\Delta_{\omega_n} \circ \cdots \circ \Delta_{\omega_1}$ gives information on the singularities over the point $\omega_1 + \cdots + \omega_n$.

The point of view on Resurgence theory that we have indicated is rather restrictive and we refer the interested reader to [Eca81, Eca93, CNP93, BSSV98] for further properties and more general definitions.

c) Bridge equation

The so-called *Bridge equation* (26) is an example of a general phenomenon which is at the origin of the name *Resurgent function*: the definition of a general resurgent function $\tilde{\varphi}$ *a priori* does not force any relationship between $\tilde{\varphi}$ and its alien derivatives, but for the resurgent functions of natural origin (*i.e.* solutions of some analytic problem) it is observed that the alien derivatives obey particular relations depending on the problem under consideration.

The equation (26) can be viewed as a bridge between alien calculus and ordinary differential calculus in the case of the formal solution $\tilde{u}$, hence its name. The families of complex numbers $A_{\omega||n}$ and $B_{\omega||n}$ which determine the differential operator in the right-hand side represent all the "transcendental" part of the information that is needed to describe the singular structure in the Borel plane (whereas the series $\tilde{u}_n$ themselves represent the "elementary" part).

In our case, the realness of the coefficients of the series $\tilde{u}_n$ implies that

$$\forall \omega \in 2\pi i\mathbb{Z}^*, \quad A_{-\omega}(b) = -\overline{A_\omega(\bar{b})},\ B_{-\omega}(b) = -\overline{B_\omega(\bar{b})},$$

and since $\tilde{u}(z,b)$ is even with respect to z, we have also

$$\forall \omega \in 2\pi i\mathbb{Z}^*, \quad A_{-\omega}(b) = A_\omega(b),\ B_{-\omega}(b) = -B_\omega(b).$$

Therefore we conclude that

$$\forall \omega \in 2\pi i\mathbb{Z}^*,\ \forall n \in \mathbb{N}, \quad A_{\omega||n} = A_{-\omega||n} \in i\mathbb{R},\ B_{\omega||n} = -B_{-\omega||n} \in \mathbb{R}.$$

The Bridge equation (26) provides the decomposition of

$$\Delta_\omega \tilde{u}(z,b) = \sum_{n\geq 0} b^n \Delta_\omega \tilde{u}_n(z)$$

as sum of its even part $A_\omega(b)\partial_b\tilde{u}(z,b)$ and its odd part $B_\omega(b)\partial_z\tilde{u}(z,b)$ (even and odd with respect to z). Note that all the successive alien derivatives may be computed by iteration of the Bridge equation:

$$\Delta_{\omega_1}\circ\Delta_{\omega_2}\tilde{u}(z,b) = \Big(A_{\omega_2}(b)\partial_b + B_{\omega_2}(b)(\partial_z-\omega_2)\Big)\Big(A_{\omega_1}(b)\partial_b + B_{\omega_1}(b)\partial_z\Big)\tilde{u}(z,b)$$

(beware of the inversion of indices), *etc.* Therefore, in principle, the singularities of each determination of the minors $\hat{u}_n$ can be expressed in terms of the numbers A_ω and B_ω and of the coefficients of the series $\tilde{u}_n$.

d) First singularity of the first minor

Here are the resurgence relations for the first series:

$$\forall\omega\in 2\pi\mathrm{i}\mathbb{Z}^*,\quad \Delta_\omega\tilde{u}_0 = A_{\omega||0}\tilde{u}_1 + B_{\omega||0}\frac{d\tilde{u}_0}{dz}.$$

On the other hand, in the case of $\omega = 2\pi\mathrm{i}$, we can rephrase Theorem 3:

$$\Delta_{2\pi\mathrm{i}}\tilde{u}_0(z) = \frac{2\pi\mathrm{i}A_5}{4!}z^4 + \frac{2\pi\mathrm{i}A_3}{2!}z^2 + \frac{2\pi\mathrm{i}A_1}{0!} + \tilde{h}(z),$$

with $\tilde{h}(z) = \sum_{p\geq 1} p!h_p z^{-p-1}$.

If we compare those two results by separating the even part from the odd part in the last identity, we obtain

$$\begin{aligned} A_{2\pi\mathrm{i}||0}\,\tilde{u}_1(z) &= 84A_{2\pi\mathrm{i}||0}\sum_{k\geq -2} d_k z^{-2k} \\ &= \frac{2\pi\mathrm{i}A_5}{4!}z^4 + \frac{2\pi\mathrm{i}A_3}{2!}z^2 + \frac{2\pi\mathrm{i}A_1}{0!} + \sum_{k\geq 1}(2k-1)!h_{2k-1}z^{-2k} \\ B_{2\pi\mathrm{i}||0}\frac{d\tilde{u}_0}{dz}(z) &= B_{2\pi\mathrm{i}||0}\sum_{k\geq 1} b_k z^{-2k-1} = \sum_{k\geq 1}(2k)!h_{2k}z^{-2k-1}, \end{aligned}$$

i.e. we find again the relations of the corollary of Theorem 4. In particular,

$$A_{2\pi\mathrm{i}||0} = \frac{1}{84}\Theta,\quad B_{2\pi\mathrm{i}||0} = \mu.$$

A.4 Idea of the proof of Theorem 6

– We already know, from Section 4, that the minor $\hat{u}_0$ of the first formal series $\tilde{u}_0$ converges at the origin and extends analytically to $\mathcal{R}^{(1)}$.

The same is true for the minors of φ_1 and φ_2. Indeed, in the case of φ_1, the relation (23) can be translated into a relation between the minors: $\hat{\varphi}_1(\zeta) = -\zeta\hat{u}_0(\zeta)$, which shows that $\hat{\varphi}_1$ extends analytically to $\mathcal{R}^{(1)}$. In the case of φ_2, consider the relation (24): the series $\varphi_1(z+1)$ admits a minor $e^{-\zeta}\hat{\varphi}_1(\zeta)$ which extends analytically to $\mathcal{R}^{(1)}$, and it can be checked that the series χ which involves the multiplicative inverses of $\varphi_1(z)$ and $\varphi_1(z+1)$ has also a minor $\hat{\chi}$ which is holomorphic in $\mathcal{R}^{(1)}$; then the operator S^{-1}, which simply amounts to division by $e^{-\zeta}-1$ for the minors, and the multiplication of series preserve the property of having a minor which extends analytically to $\mathcal{R}^{(1)}$.

We then obtain by induction that, for each $n \geq 0$ the minor $\hat{u}_n$ of $\tilde{u}_n$ converges at the origin and extends analytically to $\mathcal{R}^{(1)}$. Indeed we recall that

$$\tilde{u}_n(z) = \frac{1}{2}\left(\tilde{\Phi}_n(z) + \tilde{\Phi}_n(-z)\right) + \text{const}\,\varphi_2(z)$$

with $\tilde{v}_n = -\sum_{k=1}^{n-1} \tilde{u}_k \tilde{u}_{n-k}$ and $\tilde{\Phi}_n = -\varphi_1 S^{-1}(\varphi_2 \tilde{v}_n) + \varphi_2 S^{-1}(\varphi_1 \tilde{v}_n)$, and the same arguments as above apply.

In order to prove the analyticity of the $\hat{u}_n$'s in every sheet of the Riemann surface $\mathcal{R}$, we will use the alien derivations as a tool to "propagate" analyticity from one sheet to some nearby sheets. We first define an infinite decreasing sequence $\mathrm{RES}^{(1)}, \mathrm{RES}^{(2)}, \dots$ of subalgebras of $\mathbb{C}[z][[z^{-1}]]$, whose intersection is nothing but the algebra RES of simply ramified resurgent functions. Then we will explain how one can check that the $\tilde{u}_n$'s belong to each algebra $\mathrm{RES}^{(N)}$ and therefore to RES.

– Let $\mathrm{RES}^{(1)}$ be the subspace of $\mathbb{C}[z][[z^{-1}]]$ consisting of all the Laurent series $\tilde{\varphi}$ whose minor $\hat{\varphi}$ satisfy the following properties:

- $\hat{\varphi}$ converges at the origin and extends analytically to $\mathcal{R}^{(0)}$ (the main sheet of $\mathcal{R}$);
- $\hat{\varphi}$ extends analytically also along the paths which issue from the origin and end on $(i\mathbb{R}) \setminus 2\pi i\mathbb{Z}$ without crossing the imaginary axis (this allows to define "lateral" continuations of $\hat{\varphi}$ between the singular points);
- $\hat{\varphi}$ has at worst *ramified singularities* at $2\pi i$ and $-2\pi i$, *i.e.* singularities of the form $\mathrm{pol}(\zeta^{-1}) + \frac{1}{2\pi i}\mathrm{var}(\zeta)\log\zeta$ with $\mathrm{pol}(X) \in \mathbb{C}[X]$ and $\mathrm{var}(\zeta) \in \mathbb{C}[\zeta^{-1}]\{\zeta\}$.

One can check that $\mathrm{RES}^{(1)}$ is a subalgebra of $\mathbb{C}[z][[z^{-1}]]$ which contains RES, and on which operators $\Delta_{2\pi i}, \Delta_{-2\pi i}$ may be defined as previously and still satisfy the Leibniz rule; but these operators take their values in a space of formal series which will be larger than $\mathbb{C}[z][[z^{-1}]]$ (these formal series may involve $\log z$).

Now consider the subspace $\mathrm{RES}^{(2)}$ consisting of all the elements $\tilde{\varphi}$ of $\mathrm{RES}^{(1)}$ such that $\Delta_{2\pi i}\tilde{\varphi}$ and $\Delta_{-2\pi i}\tilde{\varphi}$ belong to $\mathrm{RES}^{(1)}$, and the lateral continuations of $\hat{\varphi}$ have ramified singularities at $\pm 4\pi i$. It is stable by multiplication too, and not only $\Delta_{\pm 2\pi i} \circ \Delta_{\pm 2\pi i}$ are defined on it, but also operators $\Delta_{4\pi i}, \Delta_{-4\pi i}$ which extend the alien derivations $\Delta_{\pm 4\pi i}$ of RES and still satisfy the Leibniz rule.

Let us try to indicate the idea behind the definition of $\mathrm{RES}^{(2)}$. The condition $\tilde{\varphi} \in \mathrm{RES}^{(1)}$ implies that the minor $\hat{\varphi}$ extends to $\mathcal{R}^{(0)}$, and even until $]2\pi i, 4\pi i[$ if

$2\pi\mathrm{i}$ is circumvented to the left or to the right: let us denote by $\hat{\varphi}^+(\zeta)$ and $\hat{\varphi}^-(\zeta)$ the two corresponding determinations of $\hat{\varphi}$ at a point ζ of $]2\pi\mathrm{i}, 4\pi\mathrm{i}[$. The minor of $\Delta_{2\pi\mathrm{i}}\tilde{\varphi}$ is nothing but

$$\hat{\chi}(\zeta) = \hat{\varphi}^+(2\pi\mathrm{i}+\zeta) - \hat{\varphi}^-(2\pi\mathrm{i}+\zeta) \quad \text{for } \zeta \in]0, 2\pi\mathrm{i}[.$$

Now the condition $\Delta_{2\pi\mathrm{i}}\tilde{\varphi} \in \mathrm{RES}^{(1)}$ implies that $\hat{\chi}$ extends to $\mathcal{R}^{(0)}$, and thus $\hat{\varphi}^+$ extends to the whole half-sheet contiguous to $\mathcal{R}^{(0)}$ defined by paths which cross $]2\pi\mathrm{i}, 4\pi\mathrm{i}[$ from right to left, since we can write for the points ζ in that half-sheet

$$\hat{\varphi}^+(\zeta) = \hat{\varphi}(\dot{\zeta}) + \hat{\chi}(\dot{\zeta} - 2\pi\mathrm{i}).$$

This is the key-point: the determination of $\hat{\varphi}$ in that half-sheet may be expressed in terms of the functions $\hat{\varphi}$ and $\hat{\chi}$ in the main sheet. Similarly, $\hat{\varphi}^-$ extends to the symmetric half-sheet, according to the formula

$$\hat{\varphi}^-(\zeta) = \hat{\varphi}(\dot{\zeta}) - \hat{\chi}(\dot{\zeta} - 2\pi\mathrm{i}).$$

Therefore, our requirement on $\Delta_{2\pi\mathrm{i}}\tilde{\varphi}$, which deals with analyticity in the main sheet for its minor, can be interpreted as a property of analyticity for $\hat{\varphi}$ in other sheets of $\mathcal{R}$.

One can go on and define inductively $\mathrm{RES}^{(3)}, \mathrm{RES}^{(4)}, \dots$, by requiring at each level that all the "computable" alien derivatives of the previous level lie in $\mathrm{RES}^{(1)}$ and adding a condition about the shape of the singularity of the minor one step farther. In fact the algebra $\mathrm{RES}^{(N)}$ at a level $N \geq 1$ is characterized by the possibility of defining on it all the operators

$$\Delta_{2\pi\mathrm{i}N_\ell} \circ \cdots \circ \Delta_{2\pi\mathrm{i}N_1}, \qquad \ell \in \mathbb{N}^*, \;\; N_1, \dots, N_\ell \in \mathbb{Z}^*, \;\; |N_1| + \cdots + |N_\ell| \leq N.$$

By definition $\mathrm{RES}^{(N+1)} \subset \mathrm{RES}^{(N)}$, and $\mathrm{RES} = \bigcap_{N\geq 1} \mathrm{RES}^{(N)}$.

– We have already seen that

$$\forall n \geq 0, \quad \tilde{u}_n \in \mathrm{RES}^{(1)}.$$

The main part of the work was done in the case of $\tilde{u}_0$, and in fact the arguments of Section 4 would allow one to check that the lateral continuations of $\hat{u}_0$ have only ramified singularities.

Taking the alien derivatives at $\pm 2\pi\mathrm{i}$ of the equations that the series $\tilde{u}_n$ satisfy and using the fact that $\Delta_{\pm 2\pi\mathrm{i}} \circ P = P \circ \Delta_{\pm 2\pi\mathrm{i}}$, we obtain a system of linear equations for the series $\Delta_{\pm 2\pi\mathrm{i}}\tilde{u}_n$, which can be written as a single equation for $\Delta_{\pm 2\pi\mathrm{i}}\tilde{u} = \sum_{n\geq 0} b^n \Delta_{\pm 2\pi\mathrm{i}}\tilde{u}_n$:

$$P(\Delta_{\pm 2\pi\mathrm{i}}\tilde{u}) = -2\tilde{u}\Delta_{\pm 2\pi\mathrm{i}}\tilde{u}.$$

We have at our disposal independent solutions of this linear equation: $\partial_b \tilde{u}$ and $\partial_z \tilde{u}$, and this allows us to prove the existence of resurgence relations

$$\Delta_{\pm 2\pi \mathrm{i}} \tilde{u}_n = \sum_{n_1+n_2=n} \Big[(n_2+1) A_{\pm 2\pi \mathrm{i} || n_1} \tilde{u}_{n_2+1} + A_{\pm 2\pi \mathrm{i} || n_1} \partial_z \tilde{u}_{n_2} \Big].$$

With this, and with the help of a verification on the shape of the singularities of the minors $\hat{u}_n$ at $\pm 4\pi \mathrm{i}$, we deduce that

$$\forall n \geq 0, \quad \tilde{u}_n \in \mathrm{RES}^{(2)}.$$

We can then proceed by induction and prove that

$$\forall N \geq 1, \forall n \geq 0, \quad \tilde{u}_n \in \mathrm{RES}^{(N)},$$

by iterating the previous arguments: at each level N, applying the alien derivations $\Delta_{\pm 2\pi \mathrm{i} N}$ (which commute with P) to the equations that the series $\tilde{u}_n$ satisfy, we obtain that the series $\Delta_{\pm 2\pi \mathrm{i} N} \tilde{u}_n$ (which could involve $\log z$) satisfy linear equations from which we deduce that they are linear combinations of the $\tilde{u}_n$'s and $\partial_z \tilde{u}_n$'s:

$$\Delta_{\pm 2\pi \mathrm{i} N} \tilde{u}_n = \sum_{n_1+n_2=n} \Big[(n_2+1) A_{\pm 2\pi \mathrm{i} N || n_1} \tilde{u}_{n_2+1} + A_{\pm 2\pi \mathrm{i} N || n_1} \partial_z \tilde{u}_{n_2} \Big]$$

(thus they are Laurent series and do *not* involve $\log z$), therefore all the series $\Delta_{2\pi \mathrm{i} N_\ell} \circ \cdots \circ \Delta_{2\pi \mathrm{i} N_1} \tilde{u}_n$ lie in $\mathrm{RES}^{(1)}$, and we obtain that the series $\tilde{u}_n$ belong to $\mathrm{RES}^{(N+1)}$ by checking the shape of the singularities at $\pm 2\pi \mathrm{i}(N+1)$ of the lateral continuations of the minors. □

A.5 Final remarks

a) Analytic classification of a class of symplectic mappings

All the previous results can be generalized to the case of the equation

$$u(z+1) - 2u(z) + u(z-1) = F(u(z)) \tag{28}$$

associated to an analytic function $F(X) = -X^2 + \mathcal{O}(X^3)$.

Any such function F determines a symplectic mapping of the plane:

$$\mathcal{F} : (X, Y) \ \mapsto \ (X_1, Y_1), \quad \begin{cases} X_1 = Y + X + F(X) \\ Y_1 = X + F(X). \end{cases}$$

A particular solution $u(z)$ of (28) yields an invariant curve $(x(z) = u(z), y(z) = u(z) - u(z-1))$ for $\mathcal{F}$, in the sense that $\mathcal{F}$ maps $(x(z), y(z))$ onto $(x(z+1), y(z+1))$.

The general normalized solution $\tilde{u}(z,b)$ that one can construct in this case provides a formal conjugation

$$\left\{ \begin{array}{l} X = \tilde{u}(z,b) \\ Y = \tilde{u}(z,b) - \tilde{u}(z-1,b) \end{array} \right.$$

between $\mathcal{F}$ and the "normal form at infinity"

$$\mathcal{T} : (z,b) \mapsto (z+1,b).$$

The coefficients $A_{\omega||n}$ and $B_{\omega||n}$ which appear in the Bridge equation can now be interpreted as *analytic invariants*, *i.e.* they allow to describe the analytic classification of the mappings $\mathcal{F}$.

b) Connection formulas

We use the Laplace transform to define two families of entire functions $u_0^+, u_1^+, \dots$ and $u_0^-, u_1^-, \dots$ in the following way:

$$\forall n \geq 0, \quad u_n^{\pm}(z) = P_n(z) + \int_0^{\pm\infty} \hat{u}_n(\zeta) \mathrm{e}^{-z\zeta}\, \mathrm{d}\zeta,$$

where P_n is the polynomial part of $\tilde{u}_n$:

$$\tilde{u}_n(z) = P_n(z) + \mathcal{O}(z^{-1}).$$

Due to the general properties of Borel and Laplace transforms, $u_n^{\pm}$ admits $\tilde{u}_n$ as Gevrey-1 asymptotic expansion in any sectorial neighborhood of $\pm\infty$ of aperture strictly less than 2π.

The formal sums

$$u^{\pm}(b,z) = \sum_{n\geq 0} b^n u_n^{\pm}(z)$$

satisfy the equation (3), and we conjecture their convergence with respect to b.

The operators in the right-hand side of the Bridge equation give rise to two formal automorphisms

$$\begin{aligned}
\Phi_{\mathrm{down}} &= \exp\Big(\sum_{N\geq 1} (A_{2\pi\mathrm{i}N}(b)\mathrm{e}^{-2\pi\mathrm{i}Nz}\partial_b + B_{2\pi\mathrm{i}N}(b)\mathrm{e}^{-2\pi\mathrm{i}Nz}\partial_z)\Big),\\
\Phi_{\mathrm{down}}(b,z) &= \Big(b + \sum_{N\geq 1} \mathcal{A}_{2\pi\mathrm{i}N}(b)\mathrm{e}^{-2\pi\mathrm{i}Nz}, z + \sum_{N\geq 1} \mathcal{B}_{2\pi\mathrm{i}N}(b)\mathrm{e}^{-2\pi\mathrm{i}Nz}\Big),
\end{aligned}$$

$$\begin{aligned}
\Phi_{\mathrm{up}} &= \exp\Big(\sum_{N\geq 1} (A_{-2\pi\mathrm{i}N}(b)\mathrm{e}^{2\pi\mathrm{i}Nz}\partial_b + B_{-2\pi\mathrm{i}N}(b)\mathrm{e}^{2\pi\mathrm{i}Nz}\partial_z)\Big),\\
\Phi_{\mathrm{up}}(b,z) &= \Big(b + \sum_{N\geq 1} \mathcal{A}_{-2\pi\mathrm{i}N}(b)\mathrm{e}^{2\pi\mathrm{i}Nz}, z + \sum_{N\geq 1} \mathcal{B}_{-2\pi\mathrm{i}N}(b)\mathrm{e}^{2\pi\mathrm{i}Nz}\Big),
\end{aligned}$$

which should allow to describe the passage from u^+ to u^-. We conjecture that they are convergent (at least with respect to z) and mutually inverse, and that

$$u^-(b,z) = u^+(\Phi_{\mathrm{down}}(b,z)), \qquad u^+(b,z) = u^-(\Phi_{\mathrm{up}}(b,z)).$$

When expanded with respect to b, these relations provide exact connection formulas between the u_n^+'s and the u_n^-'s. At first order, for large negative $\operatorname{Im} z$, we find

$$\begin{aligned} u_0^+(z) &= u_0^-\left(z + B_{2\pi i||0}\mathrm{e}^{-2\pi i z} + \mathcal{O}(\mathrm{e}^{-4\pi i z})\right) \\ &\quad + A_{2\pi i||0}\mathrm{e}^{-2\pi i z} u_1^-\left(z + \mathcal{O}(\mathrm{e}^{-2\pi i z})\right) + \mathcal{O}(z^{10}\mathrm{e}^{-4\pi i z}) \\ &= u_0^-(z) + \mathrm{e}^{-2\pi i z}\Big(A_{2\pi i||0}u_1^-(z) + B_{2\pi i||0}\partial_z u_0^-(z)\Big) + \mathcal{O}(z^{10}\mathrm{e}^{-4\pi i z}), \end{aligned}$$

as already seen in Section 2.4.

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