UNIVERSITE PARIS 7 - DENIS DIDEROT
Sciences Mathématiques de Paris Centre

THESE DE DOCTORAT
Mathématiques

Thierry Combot

Non-intégrabilité algébrique et méromorphe de problèmes de n corps et de potentiels homogènes de degré -1

Thèse dirigée par Alain ALBOUY et Jacques-Arthur WEIL

Soutenue le Vendredi 7 Décembre 2012 devant un jury composé de

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Alain ALBOUY	Observatoire de Paris
Jacques-Arthur WEIL	Université de Limoges
Carles SIMO	Université de Barcelone
Alin BOSTAN	Inria Saclay
Moulay BARKATOU	Université de Limoges
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"Chercheur en mathématiques, je m'attacherai à écrire de beaux articles, de sorte que mes collègues ressentent de l'admiration devant leur perfection et qu'ils ne puissent appuyer leur critique sur aucun détail foireux. Je m'exercerai à ne compter que sur moi-même pour garantir a 100 % l'exactitude de mes énoncés et de mes démonstrations. Je mettrai toutes les virgules au bon endroit dans les énoncés. Dans un souci d'intégration dans la communauté je rendrai justice à mes collègues vivants et défunts en m'efforçant de citer toutes les sources raisonnablement liées à mon travail, et en remerciant tous les principaux contributeurs. Je questionnerai mes collègues pour collecter des références nouvelles."

Serment d'hippocrate du mathématicien, Alain Albouy

Remerciements

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Bonne lecture

Résumé

Nous cherchons tous les potentiels homogènes de degré -1 qui soient méromorphiquement intégrables au sens de Liouville. Bien qu'une classification complète soit encore hors de portée, nous connaissons déjà de nombreuses conditions nécessaires à l'intégrabilité. Notre objectif est ici ainsi de non seulement appliquer ces critères, mais aussi d'en construire d'autres plus contraignants, ce qui nous permettrait de progresser vers notre objectif de classification complète. Dans le cadre de cette thèse, on s'intéresse non seulement aux potentiels rationnels, mais aussi aux potentiels algébriques ce qui est nécessaire étant donné que nous souhaitons que notre étude inclut le problème de n corps.

Dans un premier temps, nous définissons proprement ce qu'est le système dynamique associé à un potentiel algébrique dans le champ complexe ainsi que son intégrabilité, puis nous en déduisons que le critère usuel de Morales-Ramis-Simó pour le cas méromorphe est toujours valide. Ensuite nous construisons des conditions d'intégrabilité au second ordre, ce qui renforce les critères déjà connus. En effet, le théorème de Morales-Ramis nous donne des contraintes sur les dérivées d'ordre deux du potentiel en un point de Darboux, et notre critère étendu prend aussi en compte les dérivées d'ordre trois.

Par la suite, nous continuons de raffiner ces critères d'intégrabilité dans le cas des potentiels du plan. Les conditions d'intégrabilité à un ordre arbitraire peuvent alors être calculés pour n'importe quelle famille de potentiels, mais sous une condition générique. Sans cette condition générique, nous calculons complètement les conditions d'intégrabilité à l'ordre trois, ce qui est suffisant, conjecturalement, pour traiter n'importe quelle famille de potentiels de dimension finie.

Enfin, nous appliquons ce type de résultats dans le cadre du problème de n corps. L'invariance par rotation de ce type de problème mène d'autre part à des questions d'intégrabilité restreinte, et nous montrons alors que le problème des n corps à masses égales n'est pas intégrable même en ce sens.

Abstract

We are searching all homogeneous potentials of degree -1 meromorphically integrable in the Liouville sense. Although a complete classification seems to be still out of reach, we already know several necessary conditions to integrability. Our goal is not only to apply these already existing criterions, but also to create new ones, stronger ones, which would help us to go towards our ultimate goal of complete classification. In this thesis, we are looking for not only rational potentials, but also algebraic potentials, which is necessary as we want our study to include the n body problem.

First of all, we define properly what is the associated dynamical system to an algebraic potential in the complex domain, and its integrability. Then, we conclude that the usual criterion of Morales-Ramis-Simó for the meromorphic case still holds. Then we build second order integrability conditions, which are stronger than those already known. Indeed, the Morales-Ramis Theorem gives us constraints on the second derivative of the potential at a Darboux points, and our criterion take also into account the third order derivatives.

In the following, we continue to enhance these integrability criterions in the planar potential case. The integrability conditions at any order can then be computed for any family of potentials, but under a generic condition. Without this generic condition, we compute completely the integrability conditions up to third order, which is, hypothetically, enough to deal with any finite dimensional family of potentials.

To conclude, we apply this type of results to the n body problem. The invariance by rotation of such problems lead also to questions about restricted integrability, and we then prove that the n body problem with equal masses is not integrable even in this restricted sense.

Thèse de Doctorat de Mathématiques

présentée par Thierry Combot

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et de potentiels homogènes de degré -1

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Introduction

Motivations et rappels historiques

Dans cette thèse, nous allons nous intéresser à l'intégrabilité des systèmes hamiltoniens, et plus particulièrement aux preuves de non-intégrabilité. L'origine historique de ce type de problème remonte à Newton dès qu'il a formulé la loi de la gravitation. En effet, alors que l'on peut à partir des lois de Newton calculer explicitement le mouvement de deux corps, montrer qu'il est périodique, connaitre sa position à tout temps, etc, le problème se complique énormément dès que l'on rajoute un 3ième corps. La raison en est que l'on passe d'un système intégrable à un système non intégrable. Les équations du problème de n corps, comme de nombreux systèmes mécaniques en physique, peuvent s'écrire sous une forme hamiltonienne

$$\dot{q}_i = \frac{\partial}{\partial p_i} H \qquad \dot{p}_i = \frac{\partial}{\partial q_i} H \qquad i = 1 \dots n$$

où H est une fonction de (p, q), n est le nombre de degré de liberté, et le point correspond à la dérivée par rapport au temps. L'espace naturel sur lequel vit H est le cotangent d'une variété $\mathcal{M}$ de dimension n, et donc on aura $(p, q) \in T^*\mathcal{M}$, $T^*\mathcal{M}$ étant appelé l'espace des phases (qui lui est de dimension $2n$). Dans le cas du problème de n corps, ainsi que dans tous les problèmes que nous considérerons dans cette thèse, cet hamiltonien H aura une forme spéciale, la forme "potentiel"

$$H(p, q) = T(p) + V(q)$$

où T est une forme quadratique en p non dégénérée (appelée énergie cinétique), et V est une fonction de q (appelée potentiel). De plus, nous aurons besoin d'une définition très importante, l'homogénéité

Définition 1. *Soit* $\mathbb{K} = \mathbb{R}$ *ou* $\mathbb{C}$, *et* $V : \mathbb{K}^n \longrightarrow \mathbb{K}$. *On dit que* V *est homogène de degré* $p \in \mathbb{Z}$ *si*

$$\forall (x, \lambda) \in \mathbb{K}^n \times \mathbb{K}, \quad V(\lambda x) = \lambda^p V(x)$$

Dans le cadre du problème de n corps, on a

$$V(q) = \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{\|q_i - q_j\|}$$

Le potentiel n'est pas défini partout car il possède des singularités et, dans le complexe, les racines carrées sont des fonctions mal définies. Ainsi, dans le réel, la propriété ci dessus n'est verifiée que pour x un point non singulier et $\lambda > 0$. Dans le complexe en revanche, cette propriéte est vérifiée pour tout x non singulier et $\lambda \in \mathbb{C}^*$ à condition de choisir la "bonne branche" (qui correspond au choix du signe devant les racines carrées). Rentrons maintenant dans des définitions un peu plus techniques, notamment pour voir et comprendre l'intérêt d'une telle recherche et de preuve de non-intégrabilité. Le crochet de Poisson de deux fonctions F_1, F_2 de (p, q), qui sur $\mathbb{K}^{2n}$ est égal à

$$\{F_1, F_2\} = \sum_{j=1}^{n} \frac{\partial F_1}{\partial q_j} \frac{\partial F_2}{\partial p_j} - \frac{\partial F_1}{\partial p_j} \frac{\partial F_2}{\partial q_j}$$

L'intérêt de l'intégrabilité peut être résumée dans la définition et le théorème suivant

Définition 2. *Soit M une variété symplectique de dimension $2n$ et*

$$H : M \longrightarrow \mathbb{R} \qquad H : x \mapsto H(x)$$

un Hamiltonien à n degrsé de libertés C^∞. S'il existe des intégrales premières $I_1, \dots, I_n$, c'est à dire des fonctions telles que $\{I_i, H\} = 0$ (ce qui implique que ces fonctions sont constantes sur chaque orbite) et qui satisfont les propriétés suivantes

- *Pour tout $i, j = 1 \dots n$, on a $\{I_i, I_j\} = 0$. Cette propriété sera appelée "être en involution"*
- *Si on considère la matrice jacobienne $A \in M_{n,2n}(\mathbb{R})$ donnée par*

  $$A_{i,j} = \frac{\partial I_i}{\partial x_j} \ \ i = 1 \dots n,\ j = 1 \dots 2n,$$

  l'ensemble

  $$E = \{x \in M \text{ tels que la matrice } A \text{ soit de rang } n\}$$

  est un ouvert dense dans M. Cette propriété sera appelée "indépendants presque partout".

on dit que H est intégrable au sens de Liouville (ou complètement intégrable).

Théorème 1. *(Arnold-Liouville-Mineur [6]) Soit M une variété symplectique de dimension $2n$ et*

$$H : M \longrightarrow \mathbb{R} \qquad H : x \mapsto H(x)$$

un Hamiltonien à n degrés de libertés C^∞. Si H est intégrable au sens de Liouville, alors pour tout ensemble de niveau des intégrales premières

$$M_f = \{x \in M, I_i(x) = f_i,\ i = 1 \dots n\}$$

tel que les n fontions $I_1, \dots, I_n$ soient indépendantes sur M_f (c'est-à-dire que les n 1-formes dI_i sont linéairement indépendantes en tout point de M_f) on a

- *M_f est une variété lisse, invariante par le flot hamiltonien de H.*
- *Si la variété M_f est compacte et connexe, alors elle est difféomorphe au tore de dimension n*

  $$\mathbb{T}^n = \{(\varphi_1, \dots, \varphi_n) \text{ mod } 2\pi\}$$

- *Le flot de l'hamiltonien H sur M_f correspond a un mouvement quasi-périodique sur M_f, c'est-à-dire que dans des coordonnées angulaires $\varphi = (\varphi_1, \dots, \varphi_n)$, on a*

  $$\frac{d\varphi}{dt} = \omega \qquad \omega = \omega(f)$$

- *Les équations du système hamiltonien associé à H peuvent être résolues par quadrature.*

Ce que nous dit ce théorème, c'est que si le système est intégrable, alors on peut du point de vue dynamique bien comprendre les orbites du système, tout du moins sur un ouvert dense (car l'analyse des fibres singulières, c'est à dire des valeurs critiques de I, peut être délicate). Bien évidemment, comme l'on peut s'en douter, de tels systèmes sont rares, même si l'on n'impose pas beaucoup de régularité sur les intégrales premières. De plus, on peut dans ce cas résoudre le système par quadrature. Cela permet d'exprimer explicitement les solutions (de façon compliquée tout de même en général), mais seulement dans le cas où les intégrales premières sont complètement explicites, c'est à dire si elles sont rationnelles, algébriques, voire au pire avec des fonctions spéciales. Le cas intégrable est ainsi très rare.

Dans ce travail, nous nous proposons d'établir des méthodes permettant d'étudier non pas un hamiltonien en particulier, mais plutôt des familles avec des paramètres. On souhaite alors trouver toutes les valeurs des paramètres pour lesquels l'hamiltonien est intégrable. La présence de paramètres aura alors les conséquences suivantes

- Dans ce type d'étude, on va presque systématiquement avoir des cas exceptionnels, résistants à notre méthode d'analyse. Même si ces cas s'avèrent plus difficiles du point de vue de la preuve de non-intégrabilité, c'est justement car ils ont un comportement plus régulier que les autres, et donc qu'ils ont plus de "chance" d'être intégrable.
- Avoir des paramètres, c'est s'autoriser d'ajuster des valeurs telles que les conditions d'intégrabilité que l'on va trouver soient satisfaites, se permettre des cas singuliers, etc.. Et c'est justement ce type de recherche qui permet de trouver de nouveaux cas intégrables, notamment ceux qui sont intégrables mais de façon hautement non triviale (par exemple des intégrales premières de très haut degré)

C'est pourquoi on s'intéressera à de grandes familles, possédant parfois de nombreux cas difficiles. Les cas les plus généraux sont des classifications sur des grandes familles de potentiels, comme l'ont fait Maciejewski-Przybylska dans le recherche sur les potentiels homogènes de degré $3, 4$ dans [44],[46].

Maintenant que l'on a précisé l'intérêt des preuves de non-intégrabilité, il faut maintenant définir précisément la non-intégrabilité. En effet, nous parlons de la non existence de certaines fonctions, encore faut il préciser avec soin dans quel espace on recherche ces intégrales premières dont on prouve la non existence. Les intégrales premières les plus utiles sont les intégrales premières rationelles ou algébriques, et c'est ce type d'intégrale première qui apparait le plus souvent dans les exemples concrets de potentiels intégrables. Nous verrons de plus que l'on peut étendre notre recherche aux intégrales premières méromorphes. Ainsi, nous allons étudier la non-intégrabilité méromorphe. Il y a deux raisons à cela

- Cette classe de fonctions est déjà large. Dans le chapitre 1, nous verrons que l'on peut y adjoindre les fonctions algébriques, formant ainsi un corps de fonction un peu plus gros encore, pour peu de frais. Cela est particulièrement important lorsque l'on considère des potentiels algébriques non méromorphes. La fonction racine carrée n'est pas méromorphe par exemple, et nombreux sont les potentiels qui contiennent des racines carrées dans leur expression, notamment dans le cas du problème de n corps.
- La seconde raison est que l'on ne peut pas vraiment faire mieux. Le fait de complexifier ces systèmes dynamiques permet d'utiliser des propriétés qui n'apparaissent que dans le complexe, et cela est capital pour prouver la non-intégrabilité par l'approche que l'on va présenter.

Ainsi, les potentiels et les fonctions que nous considérerons, sauf mention explicite, seront toujours des fonctions sur $\mathbb{C}$ et les potentiels seront des fonctions de $\mathbb{C}^n$ ou d'une variété algébrique complexe (dans le cas où le potentiel est algébrique par exemple). Présentons maintenant le théorème principal qui va nous permettre de prouver la non-intégrabilité

Théorème 2. *(Morales Ramis Simó [54] Theorem 2.) On considère une variété symplectique analytique complexe M de dimension $2n$, avec le crochet de Poisson défini par la forme symplectique, H un Hamiltonien analytique sur M et $\Gamma \subset M$ une orbite (non réduite à un point). Si H possède un système complet d'intégrales premières en involution, fonctionnellement indépendantes et méromorphes sur un voisinage de Γ, alors la composante de l'identité du groupe de Galois des équations variationnelles est abélienne à tout ordre.*

Ce théorème est une généralisation de Morales-Ramis [51] Théorème 7, [50] Théorème 4.1, par le fait qu'il contraint aussi le groupe de Galois pour les équations variationnelles d'ordre supérieur. Ce théorème est la version "Galois différentiel" du théorème de Ziglin sur le groupe de monodromie [9]. L'avantage est que le groupe de Galois peut être calculé de manière effective (en dimension 2 en tout cas), alors que ce n'est pas toujours le cas du groupe de monodromie. Le fait que la contrainte d'abélianité porte sur les équations variationelles à tout ordre est important pour nous car, notamment dans les chapitres 3 et 4, nous utiliserons de nombreuses fois ces équations variationelles supérieures. A partir de là, de nombreux théorèmes de non-intégrabilité ont été démontrés. Le premier d'entre eux est à propos du potentiel de Hénon-Heiles

Théorème 3. *(Ito [33]) L'hamiltonien*

$$H(p,q) = \frac{1}{2}(p_1^2 + p_2^2) - q_2^2(A + q_1) - \frac{\lambda}{3}q_1^3 \qquad \lambda \notin \left\{ \frac{12}{k(k+1)},\ k \in \mathbb{N} \right\}$$

n'est pas méromorphiquement intégrable.

Ce théorème est redémontré par Audin [7], et raffiné par Moralès [65] qui montre qu'en fait l'hamiltonien H ne peut être méromorphiquement intégrable que pour $\lambda = 1, 2, 6, 16$. On connait d'ailleurs une intégrale première supplémentaire pour $\lambda = 6$. Abordons maintenant le choix de l'orbite Γ qu'il nous faut. Pouvoir calculer le groupe de Galois explicitement est une contrainte très forte que certains chercheurs tentent de contourner. En effet, Simó et Martinez utilisent des méthodes numériques pour démontrer la non commutativité de la composante de l'identité du groupe de Galois. D'autre part, même si l'équation variationnelle peut être obtenue explicitement, le calcul du groupe de Galois est difficile, mais peut être approché par des méthodes numériques [70].

Ainsi en pratique et pour l'instant, il faut que l'orbite soit algébrique pour calculer le groupe de Galois. Une orbite algébrique est donnée par un idéal premier de $\mathbb{C}[p, q]$ de dimension 1, invariant par le flot hamiltonien. De plus il faut que cette orbite soit suffisamment simple pour que le calcul du groupe de Galois puisse être fait de façon effective. C'est ainsi que les potentiels homogènes sont particulièrement intéressants (en plus du fait qu'ils apparaissent vraiment en physique). Ils ont cette particularité qu'il existe génériquement des solutions algébriques qui sont des droites. Plus précisément, on a

Définition 3. *Soit V un potentiel homogène de degré p méromorphe en n variables. On dira que c est un point de Darboux de V (appelé aussi configuration centrale dans le cas du problème de n corps) s'il existe un $\alpha \in \mathbb{C}$ (appelé multiplicateur) tel que $V'(c) = \alpha c$. A un point de Darboux c et à un nombre complexe $E \in \mathbb{C}$, on peut associer une orbite (appelée orbite homothétique) de la forme*

$$(q_i(t), p_i(t)) = c_i(\phi(t), \dot{\phi}(t)) \quad i = 1 \dots n \qquad \textit{avec} \qquad \frac{1}{2}\dot{\phi}(t)^2 = \frac{\alpha}{p}\phi(t)^p + E$$

L'équation $V'(c) = \alpha c$ étant (dans les cas où V est algébrique) une équation algébrique, il y a souvent non seulement une mais de multiples solutions à cette équation. Avec une orbite homothétique, on peut appliquer le théorème de Morales-Ramis-Simó, qui nous donne pour les équations variationelles au premier ordre les contraintes suivantes

Théorème 4. *(Morales, Ramis [51] Corollaire 8, [53] Théorème 5.1) Soit V un potentiel homogène de degré p, méromorphe en n variables et c un point de Darboux tel que $V'(c) = pc$ (choisir $\alpha = p$ n'est qu'un choix de normalisation, et il est toujours possible pourvu que $\alpha \neq 0$). Alors l'équation variationnelle est de la forme (en choisissant l'énergie de l'orbite $E = 1$)*

$$\ddot{X} = \phi(t)^{p-2}\nabla^2 V(c)X \qquad \textit{avec} \qquad \dot{\phi(t)}^2/2 = \phi(t)^p + 1$$

où $\nabla^2 V(c)$ désigne la matrice hessienne de V en c. Si la matrice hessienne est diagonalisable, alors la composante neutre du groupe de Galois de l'équation variationnelle est abélienne si et seulement si pour tout $\lambda \in Sp(\nabla^2 V(c))$ (le spectre de la matrice hessienne), le couple (p, λ) appartient à la table suivante

p	λ	p	λ
$\mathbb{Z}^*$	$\frac{1}{2}\, ip\,(ip+p-2)$	-3	$-\frac{25}{8}+\frac{1}{8}(\frac{6}{5}+6i)^2$
$\mathbb{Z}^*$	$\frac{1}{2}\,(ip+p-1)\,(ip+1)$	-3	$-\frac{25}{8}+\frac{1}{8}(\frac{12}{5}+6i)^2$
2	$\mathbb{C}$	3	$-\frac{1}{8}+\frac{1}{8}(2+6i)^2$
-2	$\mathbb{C}$	3	$-\frac{1}{8}+\frac{1}{8}(\frac{3}{2}+6i)^2$
-5	$-\frac{49}{8}+\frac{1}{8}(\frac{10}{3}+10i)^2$	3	$-\frac{1}{8}+\frac{1}{8}(\frac{6}{5}+6i)^2$
-5	$-\frac{49}{8}+\frac{1}{8}(4+10i)^2$	3	$-\frac{1}{8}+\frac{1}{8}(\frac{12}{5}+6i)^2$
-4	$-\frac{9}{2}+\frac{1}{2}(\frac{4}{3}+4i)^2$	4	$-\frac{1}{2}+\frac{1}{2}(\frac{4}{3}+4i)^2$
-3	$-\frac{25}{8}+\frac{1}{8}(2+6i)^2$	5	$-\frac{9}{8}+\frac{1}{8}(\frac{10}{3}+10i)^2$
-3	$-\frac{25}{8}+\frac{1}{8}(\frac{3}{2}+6i)^2$	5	$-\frac{9}{8}+\frac{1}{8}(4+6i)^2$

Le potentiel des n corps étant un potentiel homogène, cette table (entre autres) a permis la démonstrations de nombreux théorèmes de non-intégrabilité ([44, 46, 55, 68, 13, 69, 47, 45] entre autres). Dans le cas du problème de n corps, de nombreux résultats sur les configurations centrales sont connus, mais de nombreuses conjectures subsistent

Théorème 5. *(Lagrange [42], Moulton [56], Hampton-Moeckel [30], Albouy-Kaloshin [2]) Pour tout* $m_1, \dots, m_n > 0$*, après avoir fixé l'ordre de masses, il existe exactement une configuration centrale réelle du problème des n corps alignés.*
Pour tout $m_1, m_2, m_3 > 0$*, les configurations centrales sont exactement celles telles que*

- *Les masses* m_1, m_2, m_3 *sont alignées et la configuration peut se mettre sous la forme* $c = (-1, 0, \rho)$ *avec*

$$(m_2+m_3)+(2m_2+3m_3)\rho+(3m_3+m_2)\rho^2-$$
$$(3m_1+m_2)\rho^3-(3m_1+2m_2)\rho^4-(m_1+m_2)\rho^5=0$$

- *Les masses* m_1, m_2, m_3 *sont sur les sommets d'un triangle dont la longueur des cotés est tel quel*

$$r_{12}^3 = r_{13}^3 = r_{23}^3 = 1$$

Pour tout $m_1, m_2, m_3, m_4 > 0$*, il existe un nombre fini de configurations centrales dans le plan.*
Pour tout $m_1, m_2, m_3, m_4, m_5 > 0$*, il existe un nombre fini de configurations centrales dans le plan sauf éventuellement sur un ensemble algébrique de codimension* 2*.*

En voyant ces théorèmes, on voit que déjà la question de la finitude est difficile, la caractérisation encore plus et le seul résultat vraiment satisfaisant est pour 3 corps. Le théorème 3.1 de Pacella [58] nous donne cependant des informations précieuses sur les valeurs propres de la hessienne du potentiel au point de Darboux réel aligné, résultat qui fut peu exploité et que nous appliquerons dans le chapitre 5. Ainsi, le premier résultat de non-intégrabilité méromorphe obtenu est probablement le suivant

Théorème 6. *(Yoshida [73]) Le problème newtonien des* 3 *corps sur la droite avec masses égales n'est pas méromorphiquement intégrable.*

Un tel théorème est aussi démontré par Ziglin. Cependant, divers théorèmes de non-intégrabilité à propos du problème de n corps avaient déjà été démontrés dans un cadre différent. Notamment le théorème de Bruns

Théorème 7. *(Bruns [14]) Dans le problème newtonien de* 3 *corps dans l'espace, toute intégrale première algébrique par rapport aux positions, aux impulsions et au temps est une fonction algébrique des intégrales premières classiques: l'énergie, les trois composantes du moment cinétique, les* 6 *intégrales du mouvement rectiligne uniforme du centre de gravité.*

La démonstration originale de ce théorème contenait plusieurs erreurs, dont une notamment vue par Poincaré, et la démonstration a été entièrement retravaillée par Julliard-Tosel avec une petite généralisation

Théorème 8. *(Julliard Tosel [34]) Dans le problème newtonien de $n+1$ corps dans $\mathbb{R}^p$ avec $n \geq 2$, $1 \leq p \leq n+1$, toute intégrale première algébrique par rapport aux positions, aux impulsions et au temps est une fonction algébrique des intégrales premières classiques: l'énergie, les $p(p-1)/2$ composantes du moment cinétique, les $2p$ intégrales du mouvement rectiligne uniforme du centre de gravité.*

Dans un cadre plus général du point de vue de la régularité des intégrales premières, Painlevé propose

Proposition 1. *(Painlevé [60]) Le problème des 3 corps dans le plan en interaction newtonienne avec des masses strictement positives ne possède pas d'intégrale première supplémentaire qui soit algébrique en vitesse et qui ne soit pas une fonction des intégrales premières déjà connues qui sont l'énergie et le moment cinétique.*

Enfin, plus récemment, on trouve le théorème suivant

Proposition 2. *(Morales-Simon [55] Theorem 3.1. 3.2.) Pour tout $d \geq 2$, il n'existe pas d'intégrale première méromorphe au problème des 3 corps réduit par translation du centre de gravité en dimension d, qui soit indépendante avec les intégrales premières classiques (c'est à dire l'énergie et moments cinétiques). Le problème de n corps de masses égales en dimension $d \geq 2$ n'est pas méromorphiquement intégrable dans le sens de Liouville.*

Remarquons que la première partie de cette proposition parle non seulement la non intégrabilité méromorphe, mais aussi de la non existence d'intégrales supplémentaires. Dans les cas de problèmes de n corps colinéaires, peu de choses sont connues car ces preuves utilisent fortement le fait que l'on soit dans le plan. La classe de régularité donnée pour les intégrales premières recherchée est "méromorphe". Dans le chapitre 1, nous verrons que cela n'est totalement satisfaisant dans le cas du plan, et qu'ainsi la démonstration de la proposition 2 n'est pas totalement satisfaisante. C'est notamment parce que le potentiel lui même n'est pas méromorphe mais algébrique. Un problème plus "simple" existe, c'est le problème de n corps alignés, mais on n'a alors pas de bonnes propriétés permettant de prouver la non-intégrabilité de façon aisée, et d'ailleurs, on a dans un cadre proche des exemples de systèmes intégrables

Théorème 9. *(Calogero-Moser [16]) Le problème de n corps de masses égales sur la droite en interaction en $1/r^2$, correspondant au potentiel*

$$V = \sum_{n \geq i > j \geq 1} (q_i - q_j)^{-2}$$

est méromorphiquement intégrable.

S'il on prend des masses différentes de 1, alors la non-intégrabilité est prouvée pour le cas $(1, m, m, 1)$, $m \neq 1$, les autres cas restants ouverts.

Théorème 10. *(Julliard Tosel [34]) Le problème de 3 corps dans $\mathbb{R}^p$, $p \geq 2$, de masses $1, m, 1$, soumis à un potentiel en $1/r^2$, ne possède pas de système complet d'intégrales premières méromorphes dans les positions, les impulsions, en involution et indépendantes.*
Le problème de 4 corps sur la droite, de masses $1, m, m, 1$, soumis à un potentiel en $1/r^2$, ne possède pas de système complet d'intégrales premières méromorphes en involution.

Remarquons enfin que Maciejewski, Przybylska dans [44],[46] ont démontré des classifications de potentiels intégrables homogènes polynomiaux dans le plan pour le degré $3, 4$, en utilisant plusieurs orbites homothétiques simultanément pour obtenir des contraintes très fortes sur l'intégrabilité de tels systèmes.

Résultats

Dans le cadre de la mécanique céleste et du problème de n corps, les corps considérés interagissent selon le potentiel de Newton, et ainsi le potentiel est homogène de degré -1. Ainsi nous ne considérerons dans cette thèse que les potentiels homogènes de degré -1, et on s'y limitera même dans des problèmes plus généraux que le problème de n corps, non pas à cause de limitations intrinsèques des méthodes proposées à ce degré, mais simplement parce que le temps d'une thèse ne permet pas de tout faire, et qu'il faut donc se limiter à un ensemble "raisonnable". Nous verrons de plus par la suite que ce degré d'homogénéité possède certains avantages par rapport à d'autres, et certaines propriétés laissent d'ailleurs soupçonner qu'il y a "moins" de potentiels intégrables pour ce degré que pour les autres, ce qui nous facilitera (un peu) la tâche. Enfin, un travail est en cours pour appliquer ces méthodes à la démonstration de la non-intégrabilité du problème colinéaire de 3 et 4 corps ainsi que du problème de n corps du plan.

Le chapitre 1 de cette thèse sera tout d'abord consacré aux définitions, et à la question de la régularité des intégrales premières. En effet, il est nécessaire de proprement définir la notion de d'intégrale première méromorphe, notamment dans le cas du problème de n corps pour lequel le système est défini sur une surface algébrique complexe. Nous obtiendrons ainsi un théorème de non-intégrabilité similaire à celui de Morales-Ramis-Simo 2 pour des potentiels définis sur des surfaces algébriques.

Définition 4. *On considère des polynomes $G_1, \dots, G_s \in \mathbb{C}[q,w]$ et la surface algébrique complexe $\mathcal{S} = \{(q,w) \in \mathbb{C}^{n+s},\ (G_1(q,w), \dots, G_s(q,w)) = 0\}$. Soit V un potentiel méromorphe sur un ouvert $U \subset \mathcal{S}$. Soit J la matrice jacobienne de l'application $w \mapsto (G_1(q,w), \dots, G_s(q,w))$. On appelle lieu singulier de V l'ensemble*

$$\Sigma(V) = \{(q,w) \in U,\ \ V(q,w) \notin \mathbb{C} \text{ ou } det(J)(q,w) = 0\}$$

Théorème 11. *(voir Theorem 2 p 18) Soit V un potentiel méromorphe sur un ouvert $U \subset \mathcal{S}$ d'une surface algébrique complexe et $\Gamma \subset \mathbb{C}^n \times U$ une orbite non-stationnaire de V. Supposons que $\Gamma \not\subset \mathbb{C}^n \times \Sigma(V)$. S'il existe n intégrales premières méromorphes sur $\mathbb{C}^n \times (U \setminus \Sigma(V))$ de V, en involution et fonctionnellement indépendantes sur un voisinage de Γ, alors la composante de l'identité du groupe de Galois des équations variationnelles au voisinage de Γ est abélienne sur le corps de base des fonctions méromorphes sur $\Gamma \setminus (\mathbb{C}^n \times \Sigma(V))$.*

Ainsi, pour un potentiel défini sur un ouvert U d'une surface algébrique complexe, ce théorème permet de démontrer la non-intégrabilité avec des intégrales premières méromorphes sur $\mathbb{C}^n \times (U \setminus \Sigma(V))$. On peut ainsi préciser et justifier de nombreuses démonstrations de non-intégrabilité méromorphe [55, 47, 68, 47, 45, 13, 23]. Ainsi, le résultat de Morales-Simon [55] devient

Théorème 12. *Le problème de 3 corps en dimension d à masses strictement positives, réduit par translation du centre de gravité, ne possède pas d'intégrale première méromorphe en $p, q, r_{1,2}, r_{1,3}, r_{2,3}$ ($r_{1,2}, r_{1,3}, r_{2,3}$ étant les distances mutuelles) qui soit indépendante avec les intégrales premières classiques (énergie et moments cinétiques) et en involution avec le moment cinétique total.*

On a ajouté la petite contrainte supplémentaire "en involution avec le moment cinétique total" en plus, mais qui est en fait était un oubli dans [55], et qui toujours nécessaire comme l'on fait remarquer Maciejewski-Przybylska dans [47]. La précision porte sur le domaine de méromorphie des intégrales premières, qui peuvent être méromorphe en p, q mais aussi en les extensions algébriques $r_{1,2}, r_{1,3}, r_{2,3}$. Ajouter ces extensions algébriques est en fait tout à fait raisonnable car elles apparaissent déjà dans l'hamiltonien, qui est lui même une intégrale première (que l'on se doit au minimum de considérer dans une preuve de non-intégrabilité).

Le chapitre 2 est consacré à l'étude des équations variationnelles d'ordre 2 pour les potentiels homogènes de degré -1. Le théorème de Morales-Ramis 4 donne un critère d'intégrabilité en utilisant la première équation variationnelle. En utilisant les équations variationelles d'ordre 2, nous donnons un critère supplémentaire, technique à énoncer, mais étonamment simple à appliquer, qui raffine celui de Morales-Ramis

Théorème 13. *(Théorème 6 p 24) Soit V un potentiel homogène méromorphe (sauf éventuellement en 0) de degré -1, c un point de Darboux V de multiplicateur -1. Supposons que la hessienne $\nabla^2 V(c)$ de V en c, est diagonalisable. On note $\lambda_1,\dots,\lambda_n$ ses valeurs propres et $X_1,\dots,X_n$ ses vecteurs propres. On suppose que les conditions de Morales-Ramis-Yoshida 4 sont satisfaites, c'est-à-dire*

$$\lambda_i = \frac{1}{2}(n_i-1)(n_i+2),\ n_i \in \mathbb{N}, \quad i = 1\dots n$$

On peut construire un ensemble d'indices $J \subset \mathbb{N}^3$, ne dépendant que des n_i, tel que si V est méromorphiquement intégrable, alors

$$\forall (i,j,k) \in J,\ D^3V(c).(X_i,X_j,X_k) = 0$$

La construction précise de J est donnée page 24 et les suivantes. Le critère d'intégrabilité à l'ordre 2 est particulièrement simple. Il suffit de vérifier que certaines dérivées d'ordre 3 du potentiel selon les vecteurs propres de $\nabla^2 V(c)$ sont nulles. Pour l'établir, nous ramenons le calcul de commutateurs de monodromie (une intégrale de chemin) à un calcul de résidu sur les solutions de la première équation variationnelle.

Le lemme 5 donne un lien explicite entre l'abélianité du groupe de Galois et la non-nullité d'un résidu. En utilisant des algorithmes de création télescopique sur les fonctions D-finies [40],[39],[75], nous obtenons une formule explicite pour ce résidu

$$C_{i,j,k} = \begin{cases} \lim\limits_{\epsilon\to 0} \frac{3}{4\pi} f(i+\epsilon, j+\epsilon, k+\epsilon) & \text{si } i+j+k \bmod 2 = 1 \\ \lim\limits_{\epsilon\to 0} \frac{\pi}{16}\frac{1}{\Gamma(\epsilon)} f(i+\epsilon, j+\epsilon, k+\epsilon) & \text{si } i+j+k \bmod 2 = 0 \end{cases}$$

avec

$$f(i,j,k) = \frac{2^d i! j! k! \Gamma\left(\frac{1}{2}(d+1)\right)\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{c}{2}\right)}{\Gamma\left(\frac{1}{2}(a+3)\right)\Gamma\left(\frac{1}{2}(b+3)\right)\Gamma\left(\frac{1}{2}(c+3)\right)\Gamma\left(\frac{1}{2}(d+4)\right)}$$

et $a = -i+j+k$, $b = i-j+k$, $c = i+j-k$, $d = i+j+k$. Cette formule bien que compliquée peut être "devinée" par gfun [66], puis démontrée par un algorithme de création télescopique [40]. La non-nullité de ce commutateur implique la non-intégrabilité et correspond à la table A. Enfin, nous illustrons ce critère sur quelques exemples classiques.

Le chapitre 3 est consacré aux potentiels homogènes de degré -1 dans le plan. L'objet de ce chapitre est de déterminer (presque) tous les potentiels intégrables homogènes de degré -1 du plan. Pour ce faire, nous approfondissons les techniques du chapitre 2 pour les généraliser aux équations variationnelles à tout ordre. On obtient alors

Théorème 14. *(Théorème 11 p 48 et conjecture 2 p 71) Soit V un potentiel homogène du plan de degré -1 (différent de 0) tel que*

- *$V = r^{-1}U(\theta)$ en coordonnées polaires, avec U méromorphe 2π-périodique*
- *$\exists c \in \mathbb{C}^2$ tel que $c_1^2+c_2^2 \neq 0$, $V'(c) = -c$ et $\mathrm{Sp}(\nabla^2(V)(c)) \subset]-\infty, 27[$*

Si V est méromorphiquement intégrable, alors V appartient après éventuellement rotation à l'une des familles suivantes

$$V = \frac{a}{q_1} + \frac{b}{q_2} \quad a,b \in \mathbb{C}, (a,b) \neq (0,0) \qquad V = \frac{a}{r} \quad a \in \mathbb{C}^*$$

$$V = \frac{a(q_1^2+q_2^2)}{(q_1+\epsilon i q_2)^3} + \frac{a}{q_1+\epsilon i q_2} \quad a \in \mathbb{C}^*,\ \epsilon = \pm 1 \quad \textit{Hietarinta 1987 [31]}$$

Un tel théorème nécessite en effet le calcul d'équation variationnelles à un ordre arbitraire, comme nous montre l'exemple

$$U(\theta) = (1-\cos(\theta))^n - \frac{n2^n}{(2k-1)(k+1)+1} - 2^n \qquad n,k \in \mathbb{N}^*$$

Ce potentiel est intégrable à l'ordre 2 au voisinage de tout point de Darboux, et il peut être arbitrairement "plat" au voisinage de $\theta = 0$. La démonstration de ce théorème est indirecte et en particulier non constructive, contrairement aux démonstrations de non-intégrabilité dans [44],[46]. En effet, on ne "trouve" pas ces potentiels intégrables au cours du calcul, mais dans la littérature (en particulier dans [31]). On se contente de démontrer qu'il n'y en a pas d'autre.

Pour ce faire, nous introduisons un critère de rigidité dans la partie 4.3. Sous une condition de non-dégénérescence (le fait qu'un résidu lié à la k-ième équation variationnelle, donné page 59, soit non nul), on montre qu'un potentiel méromorphiquement intégrable est uniquement déterminé par son jet d'ordre k_0 au point de Darboux (voir lemme 12). En effet, on montre que la dérivée $U^{(i+1)}(0)$ est uniquement déterminée par $U^{(i)}(0)$ si ce résidu sus-cité est non nul. Pour vérifier que ce résidu est nul ou non, nous utiliserons de nouveau la création telescopique [40] puis la recherche d'une forme close [61]. Dans le cas de la valeur propre $\lambda = 2$, on trouve ainsi la formule suivante

$$\operatorname*{Res}_{t=\infty} \ (t^2-1)^{2n+2} t^{2n+1} \left(-\frac{6t^2-4}{t^2-1} + 6t \operatorname{arctanh}\left(\frac{1}{t}\right) \right)^2 =$$
$$-\frac{\pi 27^{-n} \Gamma(2n+3)}{3456 \Gamma\left(n+\frac{7}{3}\right) \Gamma\left(n+\frac{5}{3}\right)} \sum_{k=0}^{n-1} \left(\frac{(3k+4)\,\Gamma(k+5/3)\,\Gamma(k+7/3)}{(k+1)(k+2)(2k+3)\,\Gamma\left(k+\frac{13}{6}\right)\Gamma\left(k+\frac{11}{6}\right)} \right)$$

qui permet de montrer la non-nullité pour $n \geq 1$.

Nous sommes dans un cas où la première équation variationnelle est intégrable (la composante de l'identité de son groupe de Galois est abélienne). Le critère de Morales-Ramis-Yoshida 4 nous dit que les valeurs propres de la hessienne sont de la forme $\{\frac{1}{2}(n-1)(n+2),\ n \in \mathbb{N}\}$, c'est-à-dire $\lambda_i \in \{-1, 0, 2, 5, 9, 14, 20, 27, 35, \dots\}$. Ainsi le résultat de classification du théorème 14 est donc seulement partiel à cause de la restriction $\operatorname{Sp}(\nabla^2(V)(c)) \subset]-\infty, 27[$. On présente cependant une conjecture (conjecture 2 p 71 et les suivantes) et un algorithme permettant de la démontrer, ce qui permet d'augmenter la borne 27 arbitrairement.

Si l'on veut trouver tous les potentiels intégrables dans une famille E, il serait ainsi commode d'avoir une sorte de borne uniforme $\Lambda(E)$ sur les valeurs propres de la hessienne en les points de Darboux de V (définition 10). Cette borne $\Lambda(E)$ code la difficulté de trouver tous les potentiels intégrables de la famille E, et elle correspond à une borne sur le degré des invariants des équations variationnelles au voisinage des points de Darboux.

La difficulté de trouver tous les potentiels méromorphiquement intégrables d'une famille E avec des paramètres dépend peu du nombre de paramètres mais beaucoup de $\Lambda(E)$.

Dans [46], Maciejewski-Przybylska démontrent qu'il existe une relation générique sur les valeurs propres aux points de Darboux (pour un potentiel du plan) de la forme

$$\sum_{i=1}^{n} \frac{1}{\lambda_i + a} = b \tag{1}$$

ce qui permet de borner les valeurs propres. Ainsi trouver tous les potentiels homogènes intégrables parmi une famille de potentiels rationnels est génériquement "facile". Cependant, la borne sur les solutions de (1) appartenant à la table du théorème 4 peut être très grande. En effet, s'il l'on impose seulement que $\lambda_i \in \mathbb{N}$, alors la borne est doublement exponentielle en n et atteinte [36]. La borne sur le plus petit des λ_i est de l'ordre de n/b par contre, ce qui permet de majorer Λ par une constante de taille raisonnable dans le cas générique (les cas non génériques pouvant poser de grandes difficultés). Ainsi, des problèmes semblant être très difficiles, voire inaccessibles, sont en fait probablement faisables

Conjecture 1. *Soit V un potentiel homogène du plan de degré $p \geq 6$, méromorphe et analytique réel en dehors de 0 (c'est à dire $V(\mathbb{R}^2 \setminus \{0\}) \subset \mathbb{R}$). Si V est méromorphiquement intégrable, alors V possède une intégrale première supplémentaire de degré au plus 2 en les impulsions, et s'intègre complètement dans l'un des systèmes de coordonnées suivants*

- *En coordonnées cartésiennes. Les valeurs propres aux points de Darboux de multiplicateur p sont alors 0 ou $p(p-1)$*
- *En coordonnées polaires. Les valeurs propres aux points de Darboux de multiplicateur p sont alors p*
- *En coordonnées paraboliques. Les valeurs propres aux points de Darboux de multiplicateur p sont alors $1/2(p-1)$ ou $p(p+2)$*

Bien évidemment, si l'on enlève la condition $V(\mathbb{R}^2 \setminus \{0\}) \subset \mathbb{R}$, la conjecture devient beaucoup plus difficile, du moins du point de vue de la méthode de Morales-Ramis. En effet, une des conséquences les plus immédiates de cette hypothèse est que $\Lambda \leq p$. Pour les degrés $3, 4, 5$, d'autres cas sont sans doute possibles, et la preuve sera plus difficile, en raison principalement des familles spéciales qui apparaissent dans la table d'intégrabilité.

Dans le chapitre 4 (travail avec C.Koutchan), nous considérerons une famille de potentiels "difficile". En effet, pour une famille E de potentiels méromorphes homogènes de degré -1 du plan, la borne $\Lambda(E)$ n'existe pas toujours. Dans ce cas, le chapitre 3 ne peut donc pas être utilisé, et de plus la famille de potentiels que nous allons considérer nécéssite l'étude des équations variationelles d'ordre 3. Nous allons donc approfondir les méthodes du chapitre 2 pour étudier les équations variationnelle d'ordre 3 pour les potentiels méromorphes homogènes de degré -1 du plan. Ce problème est très similaire à celui laissé ouvert à la fin de [46], et présente les même difficultés que celui de la machine d'Atwood traité par Simó-Martinez dans [49]. Leur approche est très différente car utilisant des méthodes numériques, alors que nous n'utiliserons que des méthodes algébriques. En effet, les commutateurs d'éléments de monodromie que nous allons calculer sont D-finis en les paramètres, et on peut donc utiliser les techniques de création télescopique.

Théorème 15. *(Théorèmes 17, 18 p 81) Soit V un potentiel homogène de degré -1 dans le plan. On suppose que $c = (1, 0)$ est un point de Darboux de V de multiplicateur -1. Si l'équation variationnelle au voisinage d'une orbite homothétique issue de c est intégrable à l'ordre 3, alors les conditions suivantes sont vérifiées*

$$\operatorname{Sp}\left(\nabla^2 V(c)\right) = \left\{2, \tfrac{1}{2}(p-1)(p+2)\right\} \quad \textit{pour un certain } p \in \mathbb{N}.$$

$$\left(\frac{\partial^3 V}{\partial q_1 \partial q_2^2}\right)^2 f_1(p) + \left(\frac{\partial^3 V}{\partial q_2^3}\right)^2 f_2(p) + \left(\frac{\partial^4 V}{\partial q_2^4}\right) f_3(p) = 0 \quad \textit{si } p \textit{ est pair ou}$$

$$\frac{\partial^3 V}{\partial q_2^3} = 0 \quad \textit{et} \quad \left(\frac{\partial^3 V}{\partial q_1 \partial q_2^2}\right)^2 f_1(p) + \left(\frac{\partial^4 V}{\partial q_2^4}\right) f_3(p) = 0 \quad \textit{si } p \textit{ est impair}$$

Les fonctions f_1, f_2, f_3 satisfont des récurrences linéaires à coefficients polynomiaux et elles sont approximées par les expressions suivantes (dans le cas p pair)

$$f_1(2n) = \epsilon_1(n)\left(\frac{1511011}{67108864n^2} - \frac{1511011}{134217728n^3} + \frac{31731231}{4294967296n^4}\right)$$
$$f_2(2n) = \epsilon_2(n)\left(\frac{22665165}{1073741824n^4} - \frac{22665165}{1073741824n^5} + \frac{298125}{4194304n^6}\right)$$
$$f_3(2n) = \epsilon_3(n)\left(-\frac{1740684681}{68719476736n^2} + \frac{1740684681}{137438953472n^3} - \frac{2400813907}{68719476736n^4}\right)$$

avec $|\epsilon_i(n) - 1| \leq 10^{-5}$ pour tout $n \geq 100$.

Les suites f_i proviennent de calculs de commutateurs de monodromie, sont calculés sous la forme de récurrences. Cela est la forme la plus explicite que l'on peut leur trouver, et permet une utilisation directe dans les applications.

Théorème 16. *(Théorème 19, p 82) Soit V un potentiel du plan de la forme*

$$V(r,\theta) = r^{-1}\left(a + be^{i\theta} + ce^{2i\theta} + de^{3i\theta}\right). \tag{2}$$

en coordonnées polaires. Si V est méromorphiquement intégrable, alors V appartient à l'une des familles suivantes

$$V = r^{-1}a, \quad V = r^{-1}\left(a + be^{i\theta}\right), \quad V = r^{-1}\left(ae^{i\theta} + be^{3i\theta}\right),$$
$$V = r^{-1}\left(a + be^{2i\theta}\right), \; V = r^{-1}\left(a + be^{3i\theta}\right), \; V = r^{-1}\left(a + be^{i\theta}\right)^3,$$

avec $a, b \in \mathbb{C}$.

On a pour ce problème $\Lambda(E) = \infty$, et le théorème précédent est alors nécessaire. La démonstration se ramène alors à prouver qu'une suite récurrente donnée ne s'annule pas. C'est un problème difficile bien que certains algorithmes existent [35]. Dans la partie 6 de ce chapitre, nous verrons que ces récurrences possèdent une propriété utile, qui est que leurs solutions se comportent polynomialement (avec éventuellement des logarithmes) à l'infini. Cela permet de construire un algorithme résolvant l'équation $u_n = 0, \quad n \in \mathbb{N}$ pour des récurrences possédant cette propriété, présenté dans la partie 8 de ce chapitre. Une version plus difficile du théorème 15 serait un théorème portant sur le degré d'homogénéité 4. Cela permettrait de démontrer la conjecture de [46] qui a la difficulté supplémentaire de nécessiter l'analyse de valeurs propres supplémentaires possibles pour ce degré d'homogénéité.

Conjecture 2. *On considère la famille de potentiels*

$$V = aq_1^2(q_1 + iq_2)^2 + (q_1^2 + q_2^2)^2 \qquad a \in \mathbb{C}$$

Si V est méromorphiquement intégrable, alors a appartient à un ensemble fini de valeurs (à préciser).

Remarquons qu'il subsiste tout de même des problèmes ouverts même dans le cas des potentiels du plan homogènes de degré -1. En effet, une des hypothèses de base de ces théorèmes de non-intégrabilité est l'existence d'un point de Darboux. Trouver les potentiels rationnels du plan sans points de Darboux est équivalent à résoudre:

Problème: Trouver tous les $F \in \mathbb{C}(z)$ tels que $\forall z \in \mathbb{C}, \;\; F'(z) = 0 \Rightarrow z = 0$.

Ce problème possède des solutions inattendues comme

$$F(z) = f(z^n) \text{ avec } f(z) = \int \frac{az^i}{(z-\alpha)^j} dz \quad 0 \leq i \leq j-2, \ n \in \mathbb{N}^* \ \alpha \in \mathbb{C}^*$$

Comme on le voit, c'est bien souvent des problèmes avec très peu de paramètres, voire un seul, qui sont les plus difficiles. Cela ne va pas en contradiction avec les objectifs que l'on a fixé pour la non-intégrabilité. En effet, les potentiels intégrables sont hautement non génériques, et ils ont ainsi tendance à se cacher dans les cas les plus singuliers, avec peu de points de Darboux, avec des symétries spéciales, etc... cas qui s'avèrent justement être les plus difficiles à traiter.

Le chapitre 5 étudie le problème de n corps à moment cinétique non nul. En effet, la plupart des démonstrations de non-intégrabilité du problème de n corps (dont [55, 47, 13, 1]) utilisent des orbites homothétiques. Cependant, dans le cas du problème de n corps, le potentiel est invariant par rotation et ainsi à un point de Darboux (ou configuration centrale), on peut associer une orbite Keplerienne (qui n'est pas forcément homothétique). Cela nous permet alors d'étudier l'intégrabilité du champ hamiltonien associé à un potentiel homogène invariant par rotation, restreint sur à un niveau du moment cinétique (qui est une intégrale première). Dans le cas du problème de n corps à masses égales, on démontre en particulier

Théorème 17. *(Théorème 32, p 105) Le problème des n corps à masses égales dans le plan restreint à un niveau d'énergie et de moment cinétique fixé, pour $n \geq 3$ n'est pas méromorphiquement intégrable.*

Ce type de problème ne peut pas être traité en utilisant uniquement les orbites colinéaires, qui correspondent uniquement à un moment cinétique nul. L'équation variationnelle est alors, dans le cas le plus simple, une équation de Heun [3]. Ce n'est malheureusement pas toujours le cas, et il ne semble pas possible de se ramener dans le cas général à un calcul de groupe de Galois d'une équation différentielle d'ordre inférieur à 4, pour lequel il n'existe pas d'algorithme efficace de calcul de groupe de Galois, bien que l'application de [28] puisse être envisagée. Cela implique que l'on doit se contenter d'étudier certain cas plus faciles, dont fait partie le cas des masses égales.

Nous verrons d'ailleurs dans la partie 4 de ce même chapitre que l'utilisation de ces orbites elliptiques peut donner des résultats importants sur le bornage $\Lambda(E)$ des valeurs propres. En effet, l'équation de Heun possède 4 singularités régulières, et cela impose une condition supplémentaire sur les paramètres pour que le groupe de Galois ait une composante de l'identité abélienne. Ainsi, pour une orbite Keplerienne, le critère d'intégrabilité au premier ordre de Morales-Ramis 4 $\lambda \in \{\frac{1}{2}(n-1)(n+2),\ n \in \mathbb{N}\}$ devient $\lambda \in \{-1, 0\}$. On peut donc s'attendre à ce qu'il y ait beaucoup moins de cas intégrables à l'ordre 1, ce qui va grandement faciliter l'étude des équations variationnelles supérieures. Dans le cas de la dimension 3, en supposant de bonnes symétries sur le potentiels, on obtient même la classification suivante

Théorème 18. *On considère l'idéal*

$$I = \left\langle w_1^2 - x^2 - y^2, w_2^2 - x^2 - y^2 - z^2 \right\rangle$$

et V un potentiel méromorphe en w_1, w_2, z^2 sur la variété $I^{-1}(0)$ et homogène de degré -1 en dimension 3. Supposons que $V(1,0,0) \neq 0, \infty$. Si V est meromorphiquement intégrable, alors il est de la forme

$$V = \frac{a}{w_2} \ a \in \mathbb{C}^* \qquad V = \frac{b}{w_1} \ b \in \mathbb{C}^* \tag{3}$$

La démonstration se base sur un bornage des valeurs propres de la hessienne grâce à l'étude de l'équation de Heun (chapitre 5 partie 2), puis de l'application du chapitre 3.

Le chapitre 1 correspond à l'article [21], soumis à Celestial Mechanics and Dynamical Astronomy. Le chapitre 2 provient de l'article [20], accepté à Non-linearity. Le chapitre 3 provient de l'article [19], non encore soumis. Le chapitre 4 provient de l'article [22], publié dans Journal of Mathematical Physics. Enfin, le chapitre 5 provient de l'article [23], publié dans Celestial Mechanics and Dynamical Astronomy. Du fait que ces chapitres proviennent d'articles indépendants, un certain nombre de définitions standards sont répétées d'un chapitre à l'autre. Les chapitres peuvent donc, pour l'essentiel, être lus indépendamment, bien que certains résultats fassent appel à des théorèmes des chapitres précédents.

Chapter 1

Regularity questions

1.1 Introduction

The purpose of this chapter is to apply the following Theorem for proving "meromorphic" non-integrability of algebraic potentials

Theorem 1. *(Morales-Ramis-Simó [54] Theorem 2.) Let us consider a symplectic analytical complex manifold M of dimension $2n$, with the Poisson bracket defined by the symplectic form, H a Hamiltonian analytic on M and $\Gamma \subset M$ a particular (not a point) orbit. If H possesses a complete system of first integrals in involution, functionally independent and meromorphic on a neigbourhood of Γ, then the identity component of the Galois group of variational equations is abelian at any order.*

This Theorem is an extension of Theorem 7 of [51]. There is an application for homogeneous potentials in [53]. In particular, this last Theorem is directly applied to celestial mechanics problems like in [55, 68, 47, 13] where the potential is algebraic. Still, the Theorem requires explicitly that the potential should be meromorphic. Clearly an algebraic potential on $\mathbb{C}^n$ cannot be meromorphic on $\mathbb{C}^n$ unless it is rational. The problem is even worse than just being not regular enough because such potential is in fact multivalued (and thus is not a function, even on a small open set). In many other articles, the problem is either ignored or not suitably analyzed, as in [68, 69, 76]. This makes these proofs ambiguous as both the dynamical system and the notion of meromorphic integrability are not well defined.

Still Theorem 1 could be used if we consider such algebraic potential as a rational function on an algebraic complex manifold M instead of $\mathbb{C}^n$. The aim of this note is to give a clear definition of an algebraic potential, the associated dynamical system and what is a "meromorphic first integral" in this case. This will give some precisions on several non-integrability proofs in celestial mechanics, in particular Theorem 10 of [23]. A typical example is the following

$$V(q_1, q_2) = (q_1^2 + q_2^2)^{3/2}$$

On $\mathbb{C}^2$, this expression is not meromorphic, as it is not even single-valued. The main idea to circumvent such a problem is to introduce algebraic extensions

$$V(q_1, q_2, w_1) = w_1^3 \qquad w_1^2 - q_1^2 - q_2^2 = 0 \tag{1.1}$$

and then to see the function V as a function well defined on the 2-dimensional algebraic variety $\{(q_1, q_2, w_1) \in \mathbb{C}^3,\ \ w_1^2 - q_1^2 - q_2^2 = 0\}$. Let us now present more general statements.

We consider polynomials $G_1, \dots, G_s \in \mathbb{C}[q_1, \dots, q_n, w_1, \dots, w_s]$ and the ideal $I =< G_1, \dots, G_s >$. In the following, we will assume that I is a prime ideal and that the matrix

$$J \in M_s(\mathbb{C}[q, w]) \qquad J_{i,j} = \frac{\partial G_i}{\partial w_j}, \quad i, j = 1 \dots s$$

has a non-zero determinant modulo the ideal I. We define the associated manifold $\mathcal{S} = I^{-1}(0)$ and $\pi : \mathcal{S} \longrightarrow \mathbb{C}^n$ the projection on variables q.

A holomorphic function on a non empty open set $U \subset \mathcal{S}$ is by definition, locally the restriction of holomorphic functions on open sets $W \subset \mathbb{C}^{n+s}$ to U. A meromorphic function on U is locally a quotient h/k of two holomorphic functions h, k, with k non identically zero.

Let us now define derivations on $\mathcal{S}$. We first introduce the set

$$\Sigma(I) = \{(q, w) \in \mathcal{S},\ \ \det(J)(q, w) = 0\}$$

This set will be called the critical set and corresponds to points on S where the Jacobian matrix J of the application $w \longrightarrow (G_1, \dots, G_s)$ is not invertible. In example (1.1), we have in particular

$\Sigma(I) = \{w_1 = 0,\ q_1 \pm iq_2 = 0\}$. Remark that this set is at least of codimension one because the determinant is not zero modulo I. The manifold $\mathcal{S}$ is of dimension n, as it is the common zero of s functionally independent ($\det(J) \neq 0$) polynomials in dimension $n+s$.

Let U be a non empty open set of $\mathcal{S}$ and f a meromorphic function on U. We may now define

$$\frac{\partial f}{\partial q_k} = \partial_k f - (\partial_{n+1} f, \dots, \partial_{n+s} f)\, J^{-1} \,(\partial_k G_1, \dots, \partial_k G_s)^\intercal \tag{1.2}$$

where ∂_i denotes the derivative according to the i-th variable (the variables are $q_1, \dots, q_n$, $w_1, \dots, w_s$ in this order). These derivatives are well defined outside $\Sigma(I)$. We define moreover the critical set of V

$$\Sigma(V) = \{(q,w) \in U,\ \ V(q,w) \notin \mathbb{C}\} \cup (\Sigma(I) \cap U)$$

Definition 1. *A meromorphic potential V on an open set $U \subset \mathcal{S}$ defines the following dynamical system on $\mathbb{C}^n \times (U \setminus \Sigma(V))$*

$$\dot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \quad i = 1 \dots n \qquad \dot{w}_i = \sum_{j=1}^{s} p_j \frac{\partial w_i}{\partial q_j}, \quad i = 1 \dots s \tag{1.3}$$

Let us remark now that an algebraic potential fits this definition. Consider an algebraic function V on $\mathbb{C}^n$ and $P \in \mathbb{C}[q_1, \dots, q_n][w_1]$ a non-zero irreducible polynomial such that $P(V(q)) = 0$. The ideal $I = <P>$ on $\mathbb{C}[q_1, \dots, q_n, w_1]$ is prime because P is irreducible. The matrix J is 1×1 and its determinant is $\partial_{w_1} P$. As P is a non-zero irreducible polynomial, we have $\partial_w P \neq 0 \mod I$ otherwise P divides $\partial_w P$ which is impossible because $\partial_w P$ is non-zero and of degree lower than P. Thus $\mathcal{S} = I^{-1}(0)$ is a manifold of dimension n, and V is a meromorphic potential on $\mathcal{S}$, with $V(q,w) = w$. Recall that a meromorphic function on a complex algebraic manifold $\mathcal{S}$ is not strictly speaking a function, as it has singularities, and even indeterminate points, as for example

$$V(x,y) = \frac{xy}{x^2+y^2}$$

which is indeterminate at $(0,0)$, but still is meromorphic on $\mathbb{C}^2$ (and even rational).

Given a potential V on $\mathcal{S}$, the corresponding multivalued potential on $\mathbb{C}^n$ is given by $V(\pi^{-1}(q))$. The critical set $\Sigma(I)$ contains all ramification points of the multivalued expression $V(\pi^{-1}(q))$, and the critical set $\Sigma(V)$ contains all ramification/singular/indeterminate points of $V(\pi^{-1}(q))$. Thus defining the derivability in respect to the q_i as in (1.2), we find that

$$\Sigma(V) = \{(q,w) \in \mathcal{S},\ \ V \text{ is not } C^\infty \text{ at } (q,w)\}$$

We can now apply Theorem 1 to such potentials.

1.2 A Morales-Ramis-Simó Theorem for algebraic potentials

Theorem 2. *Let V be a meromorphic potential on an open set $U \subset \mathcal{S}$ and $\Gamma \subset \mathbb{C}^n \times U$ a non-stationary orbit of V. Suppose $\Gamma \not\subset \mathbb{C}^n \times \Sigma(V)$. If there are n first integrals meromorphic on $\mathbb{C}^n \times (U \setminus \Sigma(V))$ of V that are in involution and functionally independent over an open neighbourhood of Γ, then the identity component of Galois group of the variational equation near Γ is abelian over the base field of meromorphic functions on $\Gamma \setminus (\mathbb{C}^n \times \Sigma(V))$.*

Proof. One just needs to check that hypotheses of Theorem 1 are satisfied. We define $W = \Gamma \cap (\mathbb{C}^n \times \Sigma(V))$. These points W are singularities of the vector field (1.3). Let us remove these points by posing $\Gamma' = \Gamma \setminus W$. Remark that as the curve Γ is not contained in $\mathbb{C}^n \times \Sigma(V)$, Γ' is still a curve (so a non-stationary orbit). We now consider an open neighborhood $M \subset \mathbb{C}^n \times U$ of Γ' such that $M \cap \Sigma(V) = \emptyset$. So M is a complex manifold of dimension $2n$. We endow this manifold with the canonical symplectic structure in p, q, where the derivations in q are defined as in equation (1.2). This symplectic structure degenerates on $\Sigma(I)$, but we do not care as $M \cap \Sigma(I) = \emptyset$. Knowing that $M \cap \Sigma(V) = \emptyset$, we also know that the corresponding Hamiltonian

$$H(p,q,w) = \frac{1}{2}\sum_{i=1}^{n} p_i^2 + V(q,w)$$

has no singularities on M, and thus is holomorphic. All hypotheses of Theorem 1 are satisfied, and so Theorem 2 follows. □

□

So it is possible to readily apply Morales-Ramis Theorem for meromorphic potentials on a complex algebraic manifold. Remark that the additional hypothesis $\Gamma \not\subset \mathbb{C}^n \times \Sigma(V)$ can be important. For example, the potential

$$V(q_1, q_2, w_1) = w_1^5 + q_2^2 \qquad I = < w_1^2 - q_1 > \tag{1.4}$$

has a particular orbit given by $w_1(t) = 0, q_1(t) = 0, q_2(t) = \cos t$. We have $\Sigma(I) = \{w_1 = q_1 = 0, q_2 \in \mathbb{C}\}$. This orbit is non stationary, we could compute the variational equation, but still Theorem 2 does not apply because it is included in $\Sigma(I)$.

Let us now make some precisions about the base field on which we should compute the Galois group. In [51], it is written that the base field is the field of meromorphic functions on Γ', but in all applications, we compute Galois groups over the base field of rational functions. In page 114 of [53], they do not ignore this difficulty and remark that in case of a Fuchsian variational equation, this will still work because these two Galois groups are equal. However, no explicit proof is given, and so let us prove the following result.

Lemma 1. *Let*

$$\dot{x} = Ax \qquad A \in M_n(\mathbb{C}(t)) \tag{1.5}$$

be a regular singular differential equation (defined in 5.1.2 p 147 of [71]), $D \subset \mathbb{C}$ a discrete set and K the field of meromorphic functions on $\mathbb{C} \setminus D$. The Galois group G_1 of equation (1.5) *over the base field K is equal to its Galois group G_2 over the base field $\mathbb{C}(t)$.*

Proof. We consider the resolvant of equation (1.5) noted $x(t)$. Following Chapter 1.4 of [71], we define

$$\begin{aligned} Inv_1 &= \{P \in K[x_{1,1}, \dots, x_{n,n}, (\det((x)_{i,j=1\dots n}))^{-1}],\ \ P(t, x(t)) = 0\} \\ Inv_2 &= \{P \in \mathbb{C}(t)[x_{1,1}, \dots, x_{n,n}, (\det((x)_{i,j=1\dots n}))^{-1}],\ \ P(t, x(t)) = 0\} \end{aligned} \tag{1.6}$$

We have then by definition

$$G_1 = \{\sigma \in GL_n(\mathbb{C}), \forall P \in Inv_1,\ \ P(t, \sigma x(t)) = 0)\}$$

$$G_2 = \{\sigma \in GL_n(\mathbb{C}), \forall P \in Inv_2,\ \ P(t, \sigma x(t)) = 0\}$$

We know that $Inv_2 \subset Inv_1$ and so $G_1 \subset G_2$. Let $P \in Inv_1$ and let us consider γ a closed curve in $\mathbb{C} \setminus (D \cup \{z_1, \dots, z_p\})$ where $z_1, \dots, z_p$ are the singularities of equation (1.5), and σ the corresponding monodromy element. As the coefficients of P are meromorphic and univalued along the curve γ, we find that $P(t, \sigma x(t)) = 0$. Any curve in $\mathbb{C} \setminus \{z_1, \dots, z_p\}$ is homotopic to a curve in $\mathbb{C} \setminus (D \cup \{z_1, \dots, z_p\})$, so noting the monodromy group G_3, we have $G_3 \subset G_2$.

We now use the Schlesinger density Theorem ([71] Theorem 5.8 p 148). The Galois group G_2 is the Zariski closure of the monodromy group G_3. So we get

$$G_3 \subset G_1 \subset G_2 \qquad \overline{G}_3 = G_2$$

As G_1 is a Zariski closed group, we finally have $G_1 = G_2$. □

□

Typically, when Theorem 2 is used in applications, a parametrization ϕ of the curve is chosen, and the variational equation is computed according to this parametrization. In most examples, the variational equation obtained is with rational coefficients, regular singular, and the base field K for Galois group computations is an algebraic extension of meromorphic functions on $\mathbb{C} \setminus D$ where D is discrete (and $\phi(D)$ corresponds to singular points of V on Γ). Then in this case, using Lemma 1, the Galois group of the variational equation over $\mathbb{C}(t)$ is a finite extension of the Galois group over the base field K, and thus has the same identity component.

1.3 Application to homogeneous potentials

Definition 2. *Let V be a meromorphic potential on $\mathcal{S}$. We say that V is homogeneous if there exists $(d_1, d_2) \in \mathbb{Z}^* \times \mathbb{Z}$, $(k_1, \dots, k_s) \in \mathbb{Z}^s$ such that*

$$\forall (q,w) \in \mathcal{S}, \alpha \in \mathbb{C}^*, (\alpha^{d_1} q, \alpha^{k_1} w_1, \dots, \alpha^{k_s} w_s) \in \mathcal{S}, \quad V(\alpha^{d_1} q, \alpha^{k_1} w_1, \dots, \alpha^{k_s} w_s) = \alpha^{d_2} V(q,w)$$

The homogeneity degree of V is then d_2/d_1.

Theorem 3. *(Compare [53]) Let V be a homogeneous meromorphic potential on $\mathcal{S}$ of homogeneity degree $k \in \mathbb{Z}^*$ and $c \in \mathcal{S} \setminus (\{0\} \cup \Sigma(V))$ such that*

$$\frac{\partial}{\partial q_i} V(c) = \pi(c)_i \quad i = 1 \dots n$$

Suppose that $\nabla^2 V(c)$ (the Hessian matrix according to derivations in q) is diagonalizable. If V has n meromorphic first integrals on $\mathbb{C}^n \times (\mathcal{S} \setminus \Sigma(V))$ which are in involution and functionally independent, then for any $\lambda \in Sp(\nabla^2 V(c))$, the couple (k, λ) belongs to the table

k	λ	k	λ
$\mathbb{Z}^*$	$\frac{1}{2} i\,(ik + k - 2)$	-3	$\frac{25}{24} - \frac{1}{24}(\frac{6}{5} + 6i)^2$
$\mathbb{Z}^*$	$\frac{1}{2}\,(ik + k - 1)\,(ik + 1)\,/k$	-3	$\frac{25}{24} - \frac{1}{24}(\frac{12}{5} + 6i)^2$
2	$\mathbb{C}$	3	$-\frac{1}{24} + \frac{1}{24}(2 + 6i)^2$
-2	$\mathbb{C}$	3	$-\frac{1}{24} + \frac{1}{24}(\frac{3}{2} + 6i)^2$
-5	$\frac{49}{40} - \frac{1}{40}(\frac{10}{3} + 10i)^2$	3	$-\frac{1}{24} + \frac{1}{24}(\frac{6}{5} + 6i)^2$
-5	$\frac{49}{40} - \frac{1}{40}(4 + 10i)^2$	3	$-\frac{1}{24} + \frac{1}{24}(\frac{12}{5} + 6i)^2$
-4	$\frac{9}{8} - \frac{1}{4}(\frac{4}{3} + 4i)^2$	4	$-\frac{1}{8} + \frac{1}{8}(\frac{4}{3} + 4i)^2$
-3	$\frac{25}{24} - \frac{1}{24}(2 + 6i)^2$	5	$-\frac{9}{40} + \frac{1}{40}(\frac{10}{3} + 10i)^2$
-3	$\frac{25}{24} - \frac{1}{24}(\frac{3}{2} + 6i)^2$	5	$-\frac{9}{40} + \frac{1}{40}(4 + 10i)^2$

Proof. We want to use Theorem 2. As V is homogeneous, there exists $(d_1, d_2) \in \mathbb{Z}^* \times \mathbb{Z}$, $(k_1, \dots, k_p) \in \mathbb{Z}^s$ such that

$$V(\alpha^{d_1} q, \alpha^{k_1} w_1, \dots, \alpha^{k_s} w_s) = \alpha^{d_2} V(q,w)$$

We note $k = d_2/d_1$ the homogeneity degree of V. We now consider the curve $\Gamma \subset \mathcal{S}$ given by

$$q(t) = \phi(t)^{d_1}.\pi(c), \;\; w(t) = (c_{n+1}\phi(t)^{k_1}, \dots, c_{n+s}\phi(t)^{k_s}),$$
$$p(t) = d_1 \dot{\phi}(t)\phi(t)^{d_1 - 1}.\pi(c) \qquad \tfrac{1}{2} d_1^2 \dot{\phi}^2 \phi^{2d_1 - 2} = -\tfrac{d_1}{d_2}\phi^{d_2} + 1$$

This curve Γ is an orbit of V. The singular set $\Sigma(V)$ is a homogeneous variety (because V is homogeneous), and as $c \notin \Sigma(V)$, the points of $\Gamma \cap \Sigma(V)$ correspond to $\phi = 0$.

The variational equation at first order near the curve Γ is a linear differential equation in $X \in \mathbb{C}^{2n+s}$. At each point $(\dot{\phi}, \phi)$ of Γ, the vector X belongs to the tangent space of $\mathbb{C}^n \times \mathcal{S}$. Outside the singular points $\Sigma(V)$, the projection of this tangent space on the p, q variables is $\mathbb{C}^{2n}$. So we can project the variational equation and get a differential equation on $\mathbb{C}^{2n}$. Noting $\nabla^2 V(c)$ the $n \times n$ Hessian matrix in respect to the derivations in q, the projected first order variational equation is given by

$$\ddot{X} = -\phi(t)^{d_2 - 2d_1} \nabla^2 V(c) X$$

We now consider the parametrization of Γ by ϕ^{d_2} and thus making a variable change $z = \phi(t)^{d_2}$ in the first order variational equation. After diagonalizing the matrix $\nabla^2 V(c)$, we obtain n uncoupled hypergeometric equations in z

$$z(z-1)\frac{d^2 X_i}{dz^2} + \left(\frac{3k-2}{2k} z - \frac{k-1}{k}\right)\frac{dX_i}{dz} - \frac{\lambda_i}{2k} X_i = 0 \qquad \lambda_i \in \mathrm{Sp}(\nabla^2 V(c))$$

The hypergeometric equation is a Fuchsian equation. The base field on which we should compute the Galois group of this equation is the field K of meromorphic functions in $\phi, \dot{\phi}$ for $\phi \neq 0$, which due to the relation $\frac{1}{2} d_1^2 \dot{\phi}^2 \phi^{2d_1 - 2} = -\frac{d_1}{d_2} \phi^{d_2} + 1$ is an algebraic extension of degree $2d_2$ of the field of meromorphic functions in ϕ^{d_2} for $\phi^{d_2} \neq 0$ (the parametrization we have chosen). Using Lemma 1, the Galois group G_2 of the hypergeometric equation over the base field K has finite index (at most $2d_2$) in the Galois group G_1 over the base field $\mathbb{C}(z)$.

So if G_2 has an abelian identity component, then it is also the case for G_1. The Kimura table [37] gives all the cases where the Galois group G_1 of the hypergeometric equation has an abelian identity component, and this produces the table. □

□

Theorem 3 can thus be applied to algebraic potentials, and in particular for the n body problem in dimension $d \geq 2$

$$V = \sum_{1 \leq i < j \leq n} \frac{m_i m_j}{r_{i,j}} \qquad I = \left\langle \left(r_{i,j}^2 - \sum_{k=1}^{d} (q_{i,k} - q_{j,k})^2 \right)_{1 \leq i < j \leq n} \right\rangle$$

where $q_{i,\cdot}$ corresponds to the coordinates of body number i. The ideal I is prime for $d \geq 2$ (but not for $d = 1$!), and the critical set is

$$\Sigma(V) = \{(q, r) \in \mathcal{S},\ \exists i \neq j,\ r_{i,j} = 0\}$$

We have moreover that the phenomenon of (1.4) cannot appear. Indeed, all points of $\Sigma(V)$ are singularities of V (and not only ramification points), so we cannot choose a "bad" Darboux point (a Darboux point in $\Sigma(V)$).

In the articles [45, 47], some generalized problems with other homogeneity degrees are analyzed. For the generalized 3 body problem in [47], the authors only consider negative degrees, and so we still have that all points of $\Sigma(V)$ are singularities of V. In [45], such a problem could appear, but they smartly did not use forbidden orbits in their analysis. Thus the non-integrability proofs of [55, 47, 68, 47, 45, 13, 23] are confirmed using the regularity class for first integrals "meromorphic on $\mathbb{C}^n \times (\mathcal{S} \setminus \Sigma(V))$".

Chapter 2

Second order variational equations

2.1 Introduction

In this chapter, we study dynamical systems of the form

$$\ddot{q} = \nabla V(q)$$

where V is a meromorphic homogeneous function of degree -1 in $q_1, \dots, q_n$ and $n \in \mathbb{N}^*$. This system corresponds to a Hamiltonian system

$$H = \sum_{i=1}^{n} \frac{p_i^2}{2} - V(q) \qquad \dot{q}_i = \frac{\partial H}{\partial p_i} = p_i \qquad \dot{p}_i = -\frac{\partial H}{\partial q_i} = \frac{\partial V}{\partial q_i} \tag{2.1}$$

This type of Hamiltonian systems, corresponding to meromorphic homogeneous potentials, has already been studied a lot from the meromorphic integrability point of view thanks to Morales-Ramis-Simó Theorems [53, 54]. Using these, Morales-Ramis found an explicit and simple integrability criterion given by Theorem 5 (criterion of order 1). It is only necessary to find the solutions of an algebraic equation (the Darboux points) and then compute eigenvalues of Hessian matrices of the potential at these solutions. In the case of celestial mechanics (which are our primary concern due to the homogeneity degree -1), this algebraic equation corresponds to the equation of central configurations. As it is difficult to prove only the finiteness of solutions to this equation [62, 43, 2], to compute algebraically all solutions is often computationally to costly. Still, it is often possible to find one solution thanks to symmetry or a good parametrization of the potential. As we can only use one central configuration, it can be interesting to find additional integrability constraints and to put them in a simple form so they can be easily checked.

In this chapter, our main interest will be the study of variational equations of order 2 and their Galois group to produce the additional integrability criterion of Theorem 6 (criterion of order 2), and then the computation of the Galois group when the criterion is satisfied in Theorem 7. We find in particular that it is only necessary to check that some third order derivatives of the potential vanish. The method we will use (holonomic computations [38, 40, 39]) is not restricted to this single proof of Theorem 6, but can also be used to study the case of a non diagonalizable Hessian matrix in Corollary 1 (producing a different proof from Duval and Maciejewski [27]), and also other homogeneity degrees and variational equations of higher order (the main limitation would be computational power). Finally we prove Corollary 2 which gives an idea of the usefulness of second order variational equations and present some examples.

Definition 3. *A point $c \in \mathbb{C}^n$ is called a* Darboux point *of the dynamical system $\ddot{q} = \nabla V(q)$ when it satisfies the equation*

$$\nabla V(c) = \alpha c$$

The scalar $\alpha \in \mathbb{C}$ is called the multiplier *associated to c.*

As ∇V is homogeneous of degree -2, we can always choose $\alpha = 0, -1$; we will say that c is non degenerated *if $\alpha \neq 0$.*

A homothetic orbit Γ *associated to a Darboux point c with multiplier α is given by*

$$q(t) = \phi(t).c \qquad p(t) = \dot{\phi}(t).c, \qquad \textit{with} \quad \frac{1}{2}\dot{\phi}(t)^2 = -\frac{\alpha}{\phi(t)} + E$$

with $E \in \mathbb{C}^*$.

A homothetic orbit is an explicit particular solution of the dynamical system (2.1), and it will be used for non-integrability proofs (as it is used in [44, 46, 53]). In the following, we will only study non degenerate Darboux points, and always normalize the multiplier to -1. For the associated homothetic orbit, we will always choose $E = 1$, and thus

$$\frac{1}{2}\dot{\phi}^2 = \frac{1}{\phi} + 1$$

Remark that in the sequel, we will identify the dynamical system (2.1) and the potential. For example, we will say that c is a Darboux point of V. We will call "norm" and scalar product the expressions (see Craven [25])

$$\mid v \mid^2 = \sum_{i=1}^{n} v_i^2 \qquad < v, w > = \sum_{i=1}^{n} v_i w_i$$

even for complex v, w. We will say moreover that a matrix is orthonormal complex if its columns $X_1, \dots, X_n$ are such that

$$< X_i, X_j > = \sum_{k=1}^{n} (X_i)_k (X_j)_k = 0 \quad \forall i, j \qquad \mid X_i \mid^2 = \sum_{k=1}^{n} (X_i)_k^2 = 1 \quad \forall i$$

Theorem 4. *(Morales-Ramis-Simó [54] Theorem 2.) Let us consider a symplectic analytical complex manifold M of dimension $2n$, with the Poisson bracket defined by the symplectic form, H a Hamiltonian analytic on M and $\Gamma \subset M$ a non-stationary orbit. If H admits a complete system of first integrals in involution, functionally independent and meromorphic on a neigbourhood of Γ, then the identity component of the Galois group of variational equations is abelian at any order.*

Theorem 5. *(Morales-Ramis [53] Theorem 3) Let V be a meromorphic homogeneous potential of degree -1. Let c be a non degenerate Darboux point with multiplier -1. The identity component of the Galois group of the first order variational equation of V near the homothetic orbit associated to c is abelian if and only if the eigenvalues of the Hessian matrix $\nabla^2 V(c)$ satisfy*

$$Sp(\nabla^2 V(c)) \subset \left\{ \tfrac{1}{2}(n-1)(n+2),\ n \in \mathbb{N} \right\}.$$

The first and second order variational equation near a homothetic orbit are presented in sections 2.2.1, 2.2.2. Remark that the original statement of Theorem 2 in [54] was about an analytic Hamiltonian on a symplectic manifold but here we will only consider meromorphic homogeneous potentials of degree -1, which correspond in particular to analytic Hamiltonians on a neighbourhood of the homothetic orbit Γ. Theorem 3 in [53] is about any homogeneity degree but here in Theorem 5 we only picked up the case of degree -1. The first order condition is already known, computed by Yoshida [74] based on classification of hypergeometric functions by Kimura [37], and also found in a more general way by Morales-Ramis in [50, 51]. It has been used many times in [52, 44, 46], particularly in the n body problem in the case of homogeneity degree -1 in [55]. The Morales-Ramis-Simó Theorem holds for variational equations at any order, and so here we want to study completely the second order, and give an integrability characterization at order 2. The main theorems of this chapter are the following

Theorem 6. *Let V be a meromorphic homogeneous potential of degree -1. Let c be a non-degenerate Darboux point of V with multiplier -1. Assume that the Hessian $\nabla^2 V(c)$ of V in c is diagonalizable. Let $\lambda_1 \dots \lambda_n$ denote its eigenvalues and $X_1, \dots, X_n$ the corresponding eigenvectors. Assume that the integrability conditions of Morales-Ramis 5 are satisfied, i.e.*

$$\lambda_i = \frac{1}{2}(n_i - 1)(n_i + 2),\ n_i \in \mathbb{N}, \quad i = 1 \dots n$$

We can build a set of indexes $J \subset \mathbb{N}^3$, depending only on the n_i, such that if V is meromorphically integrable, then

$$\forall (i, j, k) \in J,\ D^3 V(c).(X_i, X_j, X_k) = 0$$

The set of indexes J is built as the following: $(i, j, k) \in J \Leftrightarrow A_{n_i, n_j, n_k} = 0$, where A is a 3 index table with values in $\{0, 1\}$, invariant by permutation, and given by

- For $i,j,k \in \mathbb{N}^*$, $A_{i,j,k} = 1$ if and only if one of the following conditions are satisfied

$$\left\{\begin{array}{c} i+j-k \geq 2 \\ i-j+k \geq 2 \\ -i+j+k \geq 2 \\ i+j+k \bmod 2 = 0 \end{array}\right. \text{ or } \left\{\begin{array}{c} -i+j+k \leq -3 \\ i+j+k \bmod 2 = 1 \end{array}\right. \text{ or }$$
$$\left\{\begin{array}{c} i-j+k \leq -3 \\ i+j+k \bmod 2 = 1 \end{array}\right. \text{ or } \left\{\begin{array}{c} i+j-k \leq -3 \\ i+j+k \bmod 2 = 1 \end{array}\right.$$

- For $i = 0$, $j,k \in \mathbb{N}^*$, $A_{0,j,k} = 1$ if and only if $\mid j-k \mid \geq 2$
- For $i = j = 0$, $A_{i,j,k} = 1$.

Theorem 7. *Let V be a meromorphic homogeneous potential of degree -1. Let c be a non-degenerate Darboux point of V with multiplier -1. Assume that the Hessian $\nabla^2 V(c)$ of V in c is diagonalizable. If V is meromorphically integrable, then the Galois group of the 2-nd order variational equation near the homothetic orbit associated to c with $E = 1$ is always isomorphic to $\mathbb{C}$ (meaning the additive group $\mathbb{C}_+$) and the Picard-Vessiot field of the 2-nd order variational equation is*

$$\mathbb{C}\left(\dot{\phi}, \ln\left(\dot{\phi}+\sqrt{2}\right) - \ln\left(\dot{\phi}-\sqrt{2}\right)\right)$$

except if one (or both) of the two following conditions are satisfied

- $D^3(V)(c)(v,v,v) \neq 0$ *with* $\nabla^2 V(c)v = -v,\ v \neq 0$
- $D^3(V)(c)(v,v,w) \neq 0$ *with* $\nabla^2 V(c)v = -v,\ \nabla^2 V(c)w = 0,\ v,w \neq 0$

and in this case the Galois group is $\mathbb{C}^2$ and the Picard-Vessiot field of the 2-nd order variational equation is

$$\mathbb{C}\left(\dot{\phi}, \ln\left(\dot{\phi}+\sqrt{2}\right), \ln\left(\dot{\phi}-\sqrt{2}\right)\right)$$

The integrability constraint, when the system of second order variational equations is well written, can be found by computing a particular monodromy commutator; this is the approach of Maciejewski-Przybylska in [44]. Here the main difficulty is that this monodromy commutator depends on parameters, the eigenvalues of the Hessian matrix $\nabla^2 V(c)$. Indeed, the first order integrability condition from Theorem 5 gives restrictions on such eigenvalues if the potential is meromorphically integrable, but still an infinite number of values remain possible. To compute this monodromy commutator, it is sufficient to study of a particular 3 index sequence, and then to find its zero and non zero entries. Of course it can be easily checked one by one (as done in section 2.3.1), but this is not enough. By chance, this 3 index sequence admits an explicit expression (7), but not easy to find (and to prove). The property behind it is that the monodromy commutator is holonomic with respect to the eigenvalue parameters, and so it satisfies a 3 index linear recurrence with polynomial coefficients (2.13). These recurrences are found and proved using the creative telescoping approach with the holonomic package of Mathematica [40] in section 2.3.2. A closed form solution can then be guessed by the gfun Maple package [66], and then its validity checked. The non-nullity can thus be easily studied because this closed form expression is a hypergeometric sequence.

2.2 Variational equations

2.2.1 First order variational equation

Let us first introduce the first order variational equation (see also [53, 54]). Near a homothetic orbit associated to a non-degenerate Darboux point c with multiplier -1, the first order variational equation is

$$\ddot{X} = \frac{1}{\phi(t)^3} \nabla^2 V(c) X$$

Remark that using the Euler relation on the homogeneous function V, we obtain the relation $\nabla^2 V(c)c = 2c$, and thus the eigenvalue 2 always belongs to the spectrum of $\nabla^2 V(c)$. Let us now assume that $\nabla^2 V(c)$ is diagonalizable.

Proposition 1. *(proved in Craven [57] Theorems 1,3) Let $A \in M_n(\mathbb{C})$ be a complex symmetric matrix, meaning $A_{i,j} = A_{j,i}$. Assume that A is diagonalizable. Then A is diagonalizable in an orthonormal complex basis.*

Thanks to this Proposition, we can always make a symplectic variable change

$$p \mapsto Pp \quad q \mapsto Pq$$

in the Hamiltonian H with P orthonormal complex such that the Hessian matrix of V is diagonal. As P is orthonormal complex, the Hamiltonian $H(Pp, Pq)$ is still of the form (2.1). So in the following, as long as the Hessian matrix $\nabla^2 V(c)$ is diagonalizable, we can after a linear variable change assume that $\nabla^2 V(c)$ is diagonal, noting it $D = \mathrm{diag}(\lambda_1, \dots, \lambda_n)$.

This small result is actually very important for the reduction of problems dealing with homogeneous potentials (additional similar properties can be found in Craven [25]). This is because the integrability status of a potential V is not changed after a rotation (even a complex one) nor dilatation. Such property allows a great simplification in classification in Maciejewski-Przybylska [44, 46]. The simplification at order 2 is even more important, because a simple criterion as in Theorem 6 is not possible if the Hessian matrix is not diagonal. As we will see, the only hypothesis of diagonalizability is very weak, especially because the system is seldom integrable at order 1 if the Hessian matrix is not diagonalizable (the conditions on the spectrum are still necessary, but not sufficient).

In this setting, the first order variational equation (VE_1) is now

$$\ddot{X} = \frac{1}{\phi(t)^3} DX$$

The base field on which the differential Galois group G_1 of this equation should be computed in Theorem 4 is the field $\mathbb{C}(\phi, \dot{\phi}) = \mathbb{C}(\dot{\phi})$ (because of the relation $\dot{\phi}^2/2 = 1/\phi + 1$). From now we denote K_1 the Picard-Vessiot field for (VE_1). The criterion of Theorem 5 is that the Galois group of (VE_1) has a Galois group whose identity component is abelian (also called virtually abelian), then

$$\lambda_i = \frac{1}{2}(n_i - 1)(n_i + 2), \; n_i \in \mathbb{N}, \; i = 1 \dots n$$

In the following, we will often use the variable change $\dot{\phi}/\sqrt{2} \longrightarrow t$ which transforms (one equation of) (VE_1) to

$$(t^2 - 1)\ddot{y} + 4t\dot{y} - (n_i - 1)(n_i + 2)y = 0$$

Let us now study more closely the solutions of this equation. Replacing the parameter n_i by i in this last equation, a basis of solutions is given by (P_i, Q_i) where P_i are polynomials and the functions Q_i can be written

$$Q_i(t) = P_i(t) \int \frac{1}{(t^2 - 1)^2 P_i(t)^2} dt$$

choosing 0 for the start point of the integral, just to fix the definition. The polynomials P_i can be generated by the formula

$$P_i(t) = \frac{1}{t^2 - 1} \frac{\partial^{i-1}}{\partial t^{i-1}} (t^2 - 1)^i$$

which gives a normalization for the dominant coefficient of P_i that we will choose for now. The functions Q_i can be written

$$Q_i(t) = \epsilon_i P_i(t)\text{arctanh}\left(\frac{1}{t}\right) + \frac{W_i(t)}{t^2-1}$$

with W_i polynomials, and ϵ_i is a real sequence, computed below. It follows that the differential Galois group G_1 of (VE_1) is the additive group $(\mathbb{C},+)$ (except for $i=0$).

Lemma 2. *We have* $\epsilon_i = 4^{-i}i(i+1)/i!^2$ *for all* $i \in \mathbb{N}^*$.

Proof. The sequence ϵ_i can be computed thanks to the formula

$$\epsilon_i = \int_C \frac{1}{(t^2-1)^2 P_i(t)^2} dt$$

with C a circle around $-1, 1$ in the direct way (because ϵ_i is the term in front of arctanh $(1/t)$ which grows by 1 along C). Using the symmetry $t \longrightarrow -t$, we just need to compute the residue in 1 for example. We have then

$$\epsilon_i = 2\frac{\partial}{\partial t}\left(\frac{1}{(t+1)^2 P_i(t)^2}\right)\Bigg|_{t=1}$$

knowing that 1 is never a root of P_i. Using a recurrence formula on the P_i

$$(4i^3+12i^2+8i)P_i + (-4ti^2-14ti-12t)P_{i+1} + (i+3)P_{i+2} = 0$$

we obtain

$$P_i(1) = 2^i(i+1)! \qquad \frac{\partial}{\partial t}P_i(t)\bigg|_{t=1} = 2^{i-2}i(i+3)(i+1)! \qquad \epsilon_i = \frac{4^{-i}i(i+1)}{i!^2}$$

which gives us the Lemma. □

So the functions Q are multivalued except for $i=0$, which is particular, because the Galois group is $\{Id\}$ instead of $\mathbb{C}$ and then all solutions are rational. Remark that the functions P_i, Q_i have lots of interesting properties. They are linked to Legendre functions (after a variable change), the polynomials P_i form a family of orthogonal polynomials (related to Jacobi polynomials) for the weight (t^2-1). The most important remark to do about these functions is that they are holonomic. Indeed, in the following we will consider the system

$$\begin{aligned}\big\{(4i^3+12i^2+8i)f_i(t) - (4ti^2+14ti+12t)f_{i+1}(t) + (i+3)f_{i+2}(t),\\ (t^2-1)f_i''(t) + 4tf_i'(t) - (i-1)(i+2)f_n(t)\big\}\end{aligned} \tag{2.2}$$

which vanishes for $f_i(t) = P_i$ and $f_i(t) = \epsilon_i^{-1}Q_i$.

2.2.2 Second order variational equation

The second order variational equation, when the Hessian matrix is diagonal, can be written

$$\ddot{X} = \frac{1}{\phi(t)^3}DX + \frac{1}{2}\frac{1}{\phi(t)^4}\begin{pmatrix} Y(t)^\intercal T_1 Y(t)\\ \cdots \\ Y(t)^\intercal T_n Y(t)\end{pmatrix} \qquad \ddot{Y} = \frac{1}{\phi(t)^3}DY \tag{2.3}$$

with D diagonal, $T_i \in M_n(\mathbb{C})$, $X = (x_1,\dots,x_n)$, $Y = (y_1,\dots,y_n)$. The matrix D is the Hessian matrix of V on the Darboux point c, and the matrices T_i, $i = 1\dots n$ are defined by $T_{i,j,k} = D^3(V)(c).(e_i, e_j, e_k)$ where $e_1, \dots e_n$ is the canonical basis of $\mathbb{C}^n$. A more detailed construction of second variational equation and higher orders can be found in [5] and a formula is given in [54] p 860. This system is non-linear; however, as explained in [54, 5], a linear variational equation can be constructed. We first produce a linear differential system whose solutions are linear combinations of $y_{i,j} = y_i y_j,\ \ i,j = 1 \dots n$, called the 2-nd symmetric power of equation $\ddot{Y} = \phi(t)^{-3} DY$. Equation (2.3) is then a linear differential systems whose unknowns are $x_i, y_{i,j},\ i,j = 1 \dots n$. The matrix of this linear differential system is

$$\begin{pmatrix} \Delta_1 & \dots & 0 & B_{1,1,1} & \dots & B_{1,n,n} & B_{1,1,2} & \dots & B_{1,n-1,n} \\ 0 & \dots & 0 & & \dots & & & \dots & \\ 0 & \dots & \Delta_n & B_{n,1,1} & \dots & B_{n,n,n} & B_{n,1,2} & \dots & B_{n,n-1,n} \\ 0 & \dots & 0 & \mathrm{Sym}^2(\Delta_1) & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & \mathrm{Sym}^2(\Delta_n) & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & \Delta_1 \otimes \Delta_2 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \Delta_{n-1} \otimes \Delta_n \end{pmatrix} \tag{2.4}$$

with

$$\Delta_i = \begin{pmatrix} 0 & 1 \\ \frac{\lambda_i}{\phi^3} & 0 \end{pmatrix} \qquad B_{i,j,j} = \begin{pmatrix} 0 & 0 & 0 \\ \frac{T_{i,j,j}}{2\phi^4} & 0 & 0 \end{pmatrix} \qquad B_{i,j,k} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ \frac{T_{i,j,k}}{2\phi^4} & 0 & 0 & 0 \end{pmatrix} \quad j < k$$

Definition 4. *We define the Galois group and Picard Vessiot field of an equation of the form (2.3) to be the Galois group and Picard-Vessiot field of the linear differential system whose matrix is (2.4).*

Using this Definition, we will systematically associate to a non-linear differential system of the form (2.3) the linear differential system whose matrix is (2.4). Let us denote G_2 the differential Galois group of the second order variational equation (2.3) and K_2 its Picard-Vessiot field. We denote by (LVE_2) the linear differential system corresponding to the second order variational equation (2.3).

2.2.3 Reduction of second order variational equations

Lemma 3. *Let A_1, A_2, A_3 be three block triangular matrices of the form*

$$A_1 = \begin{pmatrix} \alpha_1 & \beta_1 \\ 0 & \alpha_2 \end{pmatrix} \in \mathbb{C}(\dot{\phi}) \qquad A_2 = \begin{pmatrix} \alpha_1 & \beta_2 \\ 0 & \alpha_2 \end{pmatrix} \in \mathbb{C}(\dot{\phi}) \qquad A_3 = \begin{pmatrix} \alpha_1 & \beta_1 + \beta_2 \\ 0 & \alpha_2 \end{pmatrix}$$

Assume that $Gal_{diff}(\dot{X} = A_1 X)$ and $Gal_{diff}(\dot{X} = A_2 X)$ are virtually abelian. Then $Gal_{diff}(\dot{X} = A_3 X)$ is virtually abelian, and the Picard-Vessiot field is contained in the compositum of the Picard-Vessiot field of the two previous equations.

Proof. Let us consider $(w_1(t), z_1(t)), (w_2(t), z_2(t))$ the general solution (so depending on several constants) of the differentials systems $\dot{X} = A_1 X,\ \dot{X} = A_2 X$. As the matrices A_1, A_2 are triangular with the same right-down block, we can choose $z_1 = z_2$. By direct computation, we now find that $(w_1(t) + w_2(t), z_1(t))$ is the general solution of the differential system $\dot{X} = A_3 X$. Thus the Picard-Vessiot field $K^{(3)}$ of $\dot{X} = A_3 X$ is contained the compositum of Picard-Vessiot fields $K^{(1)}, K^{(2)}$ of $\dot{X} = A_1 X$ and $\dot{X} = A_2 X$. As the Galois group action is normal on $K^{(3)}$, an automorphism σ in $\mathrm{Gal}_{\mathrm{diff}}(\dot{X} = A_3 X)$ acts on $K^{(3)}$ as its restriction on $K^{(1)}, K^{(2)}$. So the Galois group $\mathrm{Gal}_{\mathrm{diff}}(\dot{X} = A_3 X)$ is virtually abelian. □

Lemma 4. *Assume (VE_1) has a virtually abelian Galois group. The Galois group of the second order variational equation (2.3) is virtually abelian if and only if for all $i,j,k = 1\dots n$ the systems (where the $T_{i,j,k}$ are the entries of the matrices T_i of (2.3))*

$$\ddot{x}_i = \frac{D_{i,i}}{\phi(t)^3} x_i + T_{i,j,k} y_j(t) y_k(t) \qquad \ddot{Y} = \frac{1}{\phi(t)^3} DY \tag{2.5}$$

have a virtually abelian Galois group. The Picard-Vessiot field of the second order variational equation (2.3) is the compositum of the Picard-Vessiot fields of all the systems (2.5). Moreover, the Galois group and Picard-Vessiot extension depend only on the nullity or non-nullity of $T_{i,j,k}$.

Proof. Assume that the Galois group of the second order variational equation (2.3) is virtually abelian. Let j be an integer in $1\dots n$. We now consider the subspace of solutions of equation (2.3) such that $y_i = 0$, $\forall i \neq j$. This corresponds to a subsystem of (LVE_2), whose matrix is

$$\begin{pmatrix} \Delta_1 & \dots & 0 & B_{1,j,j} \\ 0 & \dots & 0 & \dots \\ 0 & \dots & \Delta_n & B_{n,j,j} \\ 0 & \dots & 0 & \mathrm{Sym}^2(\Delta_j) \end{pmatrix} \tag{2.6}$$

As it is a subsystem of (LVE_2), the Picard-Vessiot field is a subfield of K_2, and the Galois group is virtually abelian. Let us now remark that the variable change $X = (x_1,\dots,x_n) \mapsto (x_1/\alpha_1,\dots,x_n/\alpha_n)$ multiplies the $T_{i,j,j}$ by α_i, and this variable change does not change the Galois group. So the Galois group of the differential system whose matrix is (2.6) depends only on the nullity or non-nullity of $T_{i,j,j}$.

For $j = 1\dots n$, we consider the linear differential systems whose matrices are

$$\begin{pmatrix} \Delta_1 & \dots & 0 & 0 & B_{1,j,j} & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & & \dots & & & \dots & \\ 0 & \dots & \Delta_n & 0 & B_{n,j,j} & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & \mathrm{Sym}^2(\Delta_1) & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & \mathrm{Sym}^2(\Delta_n) & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & \Delta_1 \otimes \Delta_2 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \Delta_{n-1} \otimes \Delta_n \end{pmatrix} \tag{2.7}$$

Their Galois group is virtually abelian, and their Picard Vessiot field is a subfield of K_2. We now substract to the terms over the diagonal of matrix (2.4) all the terms over the diagonal of matrices (2.7). We obtain a linear differential system whose matrix is

$$\begin{pmatrix} \Delta_1 & \dots & 0 & 0 & \dots & 0 & B_{1,1,2} & \dots & B_{1,n-1,n} \\ 0 & \dots & 0 & & \dots & & & \dots & \\ 0 & \dots & \Delta_n & 0 & \dots & 0 & B_{n,1,2} & \dots & B_{n,n-1,n} \\ 0 & \dots & 0 & \mathrm{Sym}^2(\Delta_1) & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & \mathrm{Sym}^2(\Delta_n) & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & \Delta_1 \otimes \Delta_2 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \Delta_{n-1} \otimes \Delta_n \end{pmatrix} \tag{2.8}$$

Thanks to Lemma 3, the Galois group of this linear differential equation is virtually abelian, and the Picard-Vessiot field is a subfield of K_2. The following non-linear equation

$$\ddot{X} = \frac{1}{\phi(t)^3} DX + \frac{1}{2}\frac{1}{\phi(t)^4} \begin{pmatrix} \sum\limits_{j\neq k} T_{1,j,k} y_j y_k \\ \dots \\ \sum\limits_{j\neq k} T_{1,j,k} y_j y_k \end{pmatrix} \qquad \ddot{Y} = \frac{1}{\phi(t)^3} DY \tag{2.9}$$

is associated with the linear equation (2.8). Now we do the same procedure for the terms in $y_j y_k$ of (2.9). Let j,k be two integers in $1\dots n$. We consider the solutions of equation (2.9) such that $y_i = 0,\ \forall i \neq j,k$. This corresponds to a subsystem of the associated linear differential equation (2.8), whose matrix is

$$\begin{pmatrix} \Delta_1 & \dots & 0 & B_{1,j,k} \\ 0 & \dots & 0 & \dots \\ 0 & \dots & \Delta_n & B_{n,j,k} \\ 0 & \dots & 0 & \Delta_j \otimes \Delta_k \end{pmatrix} \tag{2.10}$$

The matrix of the linear differential system associated to the non-linear systems (2.5) is either a submatrix of (2.6) if $j = k$, or a submatrix of

$$\begin{pmatrix} \Delta_1 & \dots & 0 & 0 & \dots & 0 & 0 & B_{1,j,k} & 0 \\ 0 & \dots & 0 & & \dots & & & \dots & \\ 0 & \dots & \Delta_n & 0 & 0 & \dots & 0 & B_{n,j,k} & 0 \\ 0 & \dots & 0 & \mathrm{Sym}^2(\Delta_1) & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & \mathrm{Sym}^2(\Delta_n) & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & \Delta_1 \otimes \Delta_2 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 & \dots & 0 & 0 & \dots & \Delta_{n-1} \otimes \Delta_n \end{pmatrix} \tag{2.11}$$

whose Galois group of the associated differential system is virtually abelian and Picard-Vessiot field a subfield of K_2 (as it is the case for the differential system associated to the matrix (2.10)). Remark again that the variable change $X = (x_1, \dots, x_n) \mapsto (x_1/\alpha_1, \dots, x_n/\alpha_n)$ multiplies the $T_{i,j,k}$ of matrix (2.11) by α_i, and this variable change does not change the Galois group. So the Galois group depends only on the nullity or non-nullity of $T_{i,j,k}$.

Conversely, if the Galois group of the systems (2.5) is virtually abelian, then the differential equations whose matrices are (2.8) and (2.11) have virtual abelian Galois group. Thus, thanks to Lemma 3, the Galois group of (2.3) is virtually abelian, and its Picard-Vessiot field is contained in the compositum of Picard-Vessiot fields of (2.8) and (2.11).

The Picard-Vessiot field of equation (2.3) is contained in the compositum of Picard-Vessiot fields of (2.5) and contains the Picard-Vessiot fields of all equations (2.5). Thus the Picard-Vessiot field of equation (2.3) is exactly the compositum of Picard-Vessiot fields of (2.5). ☐

The following Lemma will have a primary importance in the computation of monodromy. In fact, it will just be necessary to compute some sort of residue.

Lemma 5. *We consider $F \in \mathbb{C}(z_1)[z_2]$ and*

$$f(t) = F\left(t, arctanh\left(\frac{1}{t}\right)\right)$$

We consider the differential field and the Galois group

$$K = \mathbb{C}\left(t, arctanh\left(\frac{1}{t}\right), \int f dt\right) \qquad G = Gal_{diff}(K/\mathbb{C}(t))$$

If G is abelian, then

$$\frac{\partial}{\partial \alpha} \underset{t=\infty}{Res}\ F\left(t, arctanh\left(\frac{1}{t}\right) + \alpha\right) = 0 \quad \forall \alpha \in \mathbb{C} \tag{2.12}$$

Proof. First of all we recall that if the Galois group G is abelian, then so is the monodromy group, because the monodromy group is always included inside the Galois group. At infinity, $\operatorname{arctanh}(1/t)$ is smooth, as

$$\operatorname{arctanh}\left(\frac{1}{t}\right) = \frac{1}{t} + \frac{1}{3t} + O(t^{-2})$$

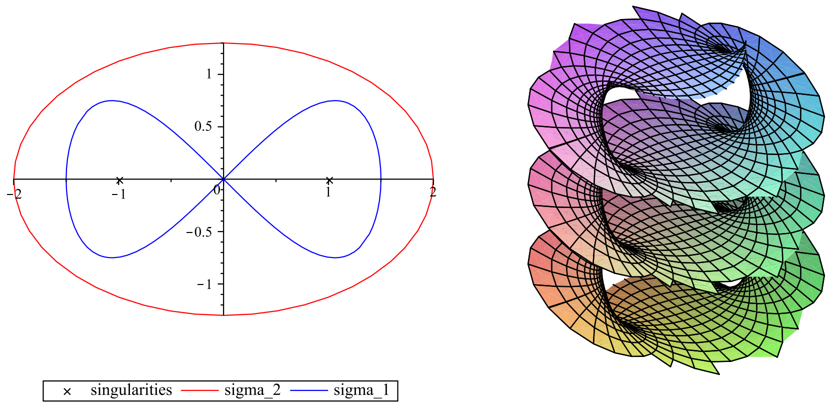


Figure 2.1: Paths corresponding to monodromy elements σ_1, σ_2, and the Riemann surface associated to $\operatorname{arctanh}(1/t)$. The difference between two sheaves is $2i\pi$ and we see that σ_2, corresponding to monodromy around infinity, acts trivially on $\operatorname{arctanh}(1/t)$.

and thus $F\left(t, \operatorname{arctanh}(1/t) + \alpha\right)$ has the following series expansion

$$\int F\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right) + \alpha\right) dt = \sum_{n=n_0}^{\infty} a_n(\alpha) t^n + r(\alpha) \ln t$$

because the function $\operatorname{arctanh}(1/t)$ is smooth at infinity. We consider two paths, the eight path σ_1 around the singularities $-1, 1$ and the path σ_2 around both. The monodromy element σ_1 fixes arctanh, and

$$\sigma_2\left(\operatorname{arctanh}\left(\frac{1}{t}\right)\right) = \operatorname{arctanh}\left(\frac{1}{t}\right) + 2i\pi$$

We consider the commutator

$$\sigma = \sigma_2^{-1} \sigma_1^{-\frac{\alpha}{2i\pi}} \sigma_2 \sigma_1^{\frac{\alpha}{2i\pi}} \quad \alpha \in 2i\pi\mathbb{Z}$$

We have $\sigma_1^{\frac{\alpha}{2i\pi}}(f) = F\left(t, \operatorname{arctanh}(1/t) + \alpha\right)$ and $\sigma_2(\ln t) = \ln t + 2i\pi$. We conclude that

$$\sigma(f) = f + r(\alpha) - r(0)$$

This $r(\alpha)$ corresponds to the residue of $F\left(t, \operatorname{arctanh}(1/t) + \alpha\right)$ at infinity. If the monodromy is abelian, then σ should act trivially on f. This is the case if and only if $r(\alpha) - r(0) = 0, \ \forall \alpha \in 2i\pi\mathbb{Z}$. The function r is polynomial in α, then $r(\alpha) - r(0) = 0$ for all α and so $r(\alpha)$ is constant. This gives the formula (2.12). □

2.2.4 Computation of terms of order 2

Let us now check that after a variable change, the $D^3(V).(X_i, X_j, X_k)$ in the integrability condition of Theorem 6 are equal to the $T_{i,j,k}$ of equation (2.3) and Lemma 4.

Proposition 2. *Let V be a meromorphic homogeneous potential of degree -1. Let $c \in \mathbb{C}^n$ be a Darboux point with multiplier -1. Assume that $\nabla^2 V(c)$ is diagonalizable. We denote its eigenvectors $X_1, \dots, X_n$. Then the integrability constraints of Theorem 6 do not depend on the choice of $X_1, \dots, X_n$. Moreover, if we make an orthonormal choice for the $X_1, \dots, X_n$, then coefficients $T_{i,j,k}$ in equation (2.3) are equal to $D^3(V).(X_i, X_j, X_k)$.*

Proof. First of all, if we make an orthonormal choice (which is always possible thanks to Proposition 1), we note P the associated orthonormal matrix. Then the potential $W(q) = V(Pq)$ has a Darboux point in $P^{-1}c$ and the corresponding Hessian matrix is diagonal. Moreover, we have

$$D^3(W).(e_i, e_j, e_k) = D^3(V).(Pe_i, Pe_j, Pe_k) = D^3(V).(X_i, X_j, X_k)$$

Now, we verify that the criterion is well defined. We first consider the case where all eigenvalues of $\nabla^2 V(c)$ are distinct. Then, up to multiplication by a non zero constant, there is a unique choice of eigenvectors $X_1, \dots, X_n$. So this does not change nullity or non-nullity of $D^3(V).(X_i, X_j, X_k)$. Now, if there are multiple eigenvalues, there are an infinite number of choices for eigenvectors X. We fix one. Assume that X_1, X_2 have the same eigenvalue. Since the nullity condition is associated only to the corresponding eigenvalues, if there is a nullity condition for some third order derivative involving X_1, it will be the same for X_2. Assume there is a condition

$$D^3(V).(X_1, X_j, X_k) = 0 \qquad D^3(V).(X_2, X_j, X_k) = 0$$

Then, for the vector $\alpha X_1 + \beta X_2$, we will also have $D^3(V).(\alpha X_1 + \beta X_2, X_j, X_k) = 0$ by expanding it. Remark that even if $j = 1, k = 1$, it will still work. All basis changes can be written as such successive linear combinations. Thus the constraints of Theorem 6 do not depend on the choice of $X_1, \dots, X_n$. □

2.3 Non integrability of second order variational equations

2.3.1 A first approach

Theorem 8. *The second order variational equation (2.3) has a virtually abelian Galois group if and only if*

- $D_{i,i} = \frac{1}{2}(p_i - 1)(p_i + 2)$ *with* $p_i \in \mathbb{N}$ *(integrability condition of order* 1*)*
- $\forall i, j, k = 1...n$, $A_{p_i, p_j, p_k} = 0 \Rightarrow T_{i,j,k} = 0$ *where* A *is a three index table with values in* $\{0, 1\}$*, invariant by permutation and whose first values are given by*

$A_{0,i,j}$	0	1	2	3	4	5	6	7	$A_{1,i,j}$	0	1	2	3	4	5	6	7
0	1	1	1	1	1	1	1	1	0	1	0	0	1	1	1	1	1
1	1	0	0	1	1	1	1	1	1	0	0	0	0	0	1	0	1
2	1	0	0	0	1	1	1	1	2	0	0	0	0	0	0	1	0
3	1	1	0	0	0	1	1	1	3	1	0	0	0	0	0	0	1
4	1	1	1	0	0	0	1	1	4	1	0	0	0	0	0	0	0
5	1	1	1	1	0	0	0	1	5	1	1	0	0	0	0	0	0
6	1	1	1	1	1	0	0	0	6	1	0	1	0	0	0	0	0
7	1	1	1	1	1	1	0	0	7	1	1	0	1	0	0	0	0
$A_{2,i,j}$	0	1	2	3	4	5	6	7	$A_{3,i,j}$	0	1	2	3	4	5	6	7
0	1	0	0	0	1	1	1	1	0	1	1	0	0	0	1	1	1
1	0	0	0	0	0	0	1	0	1	1	0	0	0	0	0	0	1
2	0	0	1	0	0	0	0	1	2	0	0	0	1	0	0	0	0
3	0	0	0	1	0	0	0	0	3	0	0	1	0	1	0	0	0
4	1	0	0	0	1	0	0	0	4	0	0	0	1	0	1	0	0
5	1	0	0	0	0	1	0	0	5	1	0	0	0	1	0	1	0
6	1	1	0	0	0	0	1	0	6	1	0	0	0	0	1	0	1
7	1	0	1	0	0	0	0	1	7	1	1	0	0	0	0	1	0
$A_{4,i,j}$	0	1	2	3	4	5	6	7	$A_{5,i,j}$	0	1	2	3	4	5	6	7
0	1	1	1	0	0	0	1	1	0	1	1	1	1	0	0	0	1
1	1	0	0	0	0	0	0	0	1	1	1	0	0	0	0	0	0
2	1	0	0	0	1	0	0	0	2	1	0	0	0	0	1	0	0
3	0	0	0	1	0	1	0	0	3	1	0	0	0	1	0	1	0
4	0	0	1	0	1	0	1	0	4	0	0	0	1	0	1	0	1
5	0	0	0	1	0	1	0	1	5	0	0	1	0	1	0	1	0
6	1	0	0	0	1	0	1	0	6	0	0	0	1	0	1	0	1
7	1	0	0	0	0	1	0	1	7	1	0	0	0	1	0	1	0

One direct application of this theorem is the study of problems in celestial mechanics, like in [55, 68]. It is often unnecessary to know the table A for arbitrary high eigenvalues, except in some cases like the open problem at the end of [46], which is, because of this, much more difficult.

Proof. Using Lemma 4, the study of the Galois group of the second order variational equation (2.3) reduces to the study of equation (2.5). Let us look first at the expressions of functions Y. The variable change $\dot{\phi}/\sqrt{2} \longrightarrow t$ gives the equation

$$\left(t^2-1\right)\ddot{y}+4t\dot{y}-(i-1)(i+2)y=0$$

for $Y_i(t)$. A basis of solutions is given by (P_i, Q_i) as presented in section 2.2.1. We obtain for $y(t)$ the following solution

$$y(t)=C_1P_i(t)+C_2Q_i(t)+$$
$$\int Y_j(t)Y_k(t)P_i(t)(t^2-1)^2dtQ_i(t)-\int Y_j(t)Y_k(t)Q_i(t)(t^2-1)^2dtP_i(t)$$

So we need to study the monodromy of the non homogeneous part of the solution. Let us try to apply Lemma 5. This theorem does not apply directly because there could be compensations between the two integrals. But we can rewrite it

$$\int Q_j(t)Q_k(t)P_i(t)(t^2-1)^2 dt P_i(t)\int \frac{1}{(t^2-1)^2P_i(t)^2}dt -$$
$$\int Q_j(t)Q_k(t)P_i(t)\int \frac{1}{(t^2-1)^2P_i(t)^2}dt(t^2-1)^2 dt P_i(t) =$$
$$\int\int Q_j(t)Q_k(t)P_i(t)(t^2-1)^2 dt \frac{1}{(t^2-1)^2P_i(t)^2}dt P_i(t)$$

Then

$$\int Q_j(t)Q_k(t)P_i(t)dt$$

is in the Picard-Vessiot field of equation (2.5) (because P_i is a polynomial and that the Picard-Vessiot field is stable by derivation). We also know that Q_i is in the Picard-Vessiot field, and then by subtraction,

$$\int Q_j(t)Q_k(t)Q_i(t)(t^2-1)^2 dt$$

is in the Picard-Vessiot field of equation (2.5). We can now apply Lemma 5 to this integral, and thus we just need to study the residue

$$S = \operatorname*{Res}_{t=\infty} (t^2-1)^2(Q_i(t)+\epsilon_i\alpha P_i)(Q_j(t)+\epsilon_j\alpha P_j)(Q_k(t)+\epsilon_k\alpha P_k)dt$$

The integrability conditions at order 2 only require that the identity component of the Galois group should be abelian. This does not say a priori anything about the whole Galois group, except if it is connected. This is the case here. The Galois group at order 1 is $\mathbb{C}$ (or Id). Then at order 2, the Picard-Vessiot field will be of the form

$$K = \mathbb{C}\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right), \int f(t)dt\right) \quad \text{with} \quad f(t) \in \mathbb{C}\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right)\right)$$

We just add some integral in K. Then the Galois group $\mathrm{Gal}_{\mathrm{diff}}(K,\mathbb{C}(t))$ is still connected.

So, we write in the table $A_{i,j,k} = 1$ if the constraint of Lemma 5 is satisfied, 0 otherwise. This criterion is a priori a necessary criterion, but not sufficient. So we then try to compute the integral using integration by parts

$$\int Q_j(t)Q_k(t)P_i(t)(t^2-1)^2 dt Q_i(t) - \int Q_j(t)Q_k(t)Q_i(t)(t^2-1)^2 dt P_i(t)$$

Using the expressions of functions Q, we need to integrate functions in $\mathbb{C}[t, \operatorname{arctanh}(1/t)]$. We make successive integrations by part, differentiating the term in $\operatorname{arctanh}(1/t)$ of the highest degree, and finally we obtain an integral of a function in $\mathbb{C}(t)$. This procedure could fail, but it works every time for $A_{i,j,k} = 1$ given in the table (for $A_{i,j,k} = 0$, the procedure fails because terms in $\ln(t^2-1)$ appear). Using Lemma 4, we know that it is necessary and sufficient that all equations

$$\ddot{X} = \frac{1}{\phi(t)^3}d_iX + \frac{1}{2}\frac{1}{\phi(t)^4}Y_j(t)Y_k(t)$$

have a virtually abelian Galois group for all non zero $T_{i,j,k}$ for the virtual abelianity of the Galois group of equation (2.3).

Theorem 9. *The table A of Theorem 8 has the following values*

- *For $i, j, k \in \mathbb{N}^*$, $A_{i,j,k} = 1$ if and only if one of the following conditions are satisfied*

$$\begin{cases} i+j-k \geq 2 \\ i-j+k \geq 2 \\ -i+j+k \geq 2 \\ i+j+k \ mod\ 2 = 0 \end{cases} \quad or \quad \begin{cases} -i+j+k \leq -3 \\ i+j+k \ mod\ 2 = 1 \end{cases} \quad or$$

$$\begin{cases} i-j+k \leq -3 \\ i+j+k \ mod\ 2 = 1 \end{cases} \quad or \quad \begin{cases} i+j-k \leq -3 \\ i+j+k \ mod\ 2 = 1 \end{cases}$$

- *For $i = 0$, $j, k \in \mathbb{N}^*$, $A_{0,j,k} = 1$ if and only if $\mid j-k \mid \geq 2$*
- *For $i = j = 0$, $A_{i,j,k} = 1$.*

Moreover, the table A is invariant by permutation of the indexes i, j, k.

This table is the direct generalization of the integrability table of [53] at order 2 for degree -1. A similar process could be done for other homogeneity degrees, but in fact the degree -1 is much simpler for three reasons

- There is only one family in the Morales Ramis table for degree -1, and generically, there are two (and the complexity increase with the power three of the number of families).
- Some homogeneity degrees have very particular families, associated to the groups A_4, S_4, A_5. This would produce exceedingly complicated computations. But this does not happen here.
- By studying only one homogeneity degree, we get one less parameter in the holonomic computations. This is important because computational cost usually increase exponentially with the number of parameters (at least).

2.3.2 General case

Using the last theorem, we already know that we just have to study the residue

$$\underset{t=\infty}{\text{Res}}\ (t^2-1)^2(Q_j(t)+\epsilon_j\alpha P_j)(Q_k(t)+\epsilon_k\alpha P_k)(Q_k(t)+\epsilon_k\alpha P_k)$$

and a necessary condition for integrability is that this residue should be independent of α. We will call the fact that the coefficient in α^2 should be zero “the constraint in α^2” and respectively “the constraint in α” for the term in α. We see also that a priori, the residue is a polynomial of degree 3 in α but we have (noting [ ] the coefficient extraction)

$$[\alpha^3]\ \underset{t=\infty}{\text{Res}}\ (Q_i(t)+\epsilon_i\alpha P_i)(Q_j(t)+\epsilon_j\alpha P_j)(Q_k(t)+\epsilon_k\alpha P_k)(t^2-1)^2 =$$

$$\underset{t=\infty}{\text{Res}}\ P_iP_jP_k\epsilon_i\epsilon_j\epsilon_k(t^2-1)^2 = 0$$

because it is a polynomial in t. So we have just two constraints for integrability. To study this residue, we will consider the holonomic system (2.2) which vanishes for $f_n(t) = P_n$ and $f_n(t) = \epsilon_n^{-1}Q_n$. The holonomic system

$$\left\{-4tf_n(t)+(t^2-1)f_n'(t), -f_n(t)+f_{n+1}(t)\right\}$$

vanishes for $f_n(t) = (t^2-1)^2$. We will use these systems to compute recurrences for our residues.

Proof. First part. We prove that the Galois group is not virtually abelian if the variational equation contains a term corresponding to indexes such that $A_{i,j,k} = 0$. We will begin with the non zero indexes case (this is because the index 0 is very special, in particular the function Q_0 is not multivalued). We now need to compute the residues of Lemma 5 for all indexes and prove they are non zero for $A_{i,j,k} = 0$. Knowing that $\epsilon_i \neq 0 \quad i \geq 1$, we only have to study of the sequence

$$S_{i,j,k} = \operatorname*{Res}_{t=\infty} (\epsilon_i^{-1} Q_i(t) + \alpha P_i)(\epsilon_j^{-1} Q_j(t) + \alpha P_j)(\epsilon_k^{-1} Q_k(t) + \alpha P_k)(t^2 - 1)^2$$

Moreover we know that the differential system we use vanishes for P_n and $\epsilon_n^{-1} Q_n$, and thus is also vanishing for $\epsilon_n^{-1} Q_n + \alpha P_n$. Thanks to that, we will be able to find a recurrence on $S_{i,j,k}$, and besides it will not depend on α.

Lemma 6. *The sequence $S_{i,j,k}$ satisfies the following recurrence relations*

$$\begin{gathered}(1+i+j-k)(k+1)(i-j+k)S_{i,j,k} - j(-2+i+j-k)(3+i-j+k)S_{i,j-1,k+1} = 0 \\ 4(i-j-k)(1+i+j-k)(k+1)(k+2)(i-j+k)(1+i+j+k)S_{i,j,k} - \\ (i-j-k-3)(i+j-k-2)(3+i-j+k)(4+i+j+k)S_{i,j,k+2} = 0\end{gathered} \tag{2.13}$$

and all the recurrences produced permuting i, j, k as well.

This recurrence relation can be proved automatically using the Mgfun package for Maple, or the holonomic package for Mathematica [30], or even by hand using integration by parts and a formula between the derivative of P_n and P_n, P_{n-1}. Let us remark that these relations conserve the parity of $i + j + k$ which will allow us to treat both cases independently. We will actually prove more than required for proving Theorem 9: we will find closed form solutions for the residue we need to compute.

Lemma 7. *For $i, j, k \in \mathbb{N}^*$, we let*

$$f(i,j,k) = \frac{2^d i! j! k! \Gamma\left(\frac{1}{2}(d+1)\right)\Gamma\left(\frac{a}{2}\right)\Gamma\left(\frac{b}{2}\right)\Gamma\left(\frac{c}{2}\right)}{\Gamma\left(\frac{1}{2}(a+3)\right)\Gamma\left(\frac{1}{2}(b+3)\right)\Gamma\left(\frac{1}{2}(c+3)\right)\Gamma\left(\frac{1}{2}(d+4)\right)}$$

with $a = -i+j+k$, $b = i-j+k$, $c = i+j-k$, $d = i+j+k$. We have then the following formula

$$\frac{\partial}{\partial \alpha} S_{i,j,k} = \begin{cases} \lim\limits_{\epsilon \to 0} \frac{3}{4\pi} f(i+\epsilon, j+\epsilon, k+\epsilon)\alpha & \text{if } i+j+k \mod 2 = 1 \\ \lim\limits_{\epsilon \to 0} \frac{\pi}{16} \frac{1}{\Gamma(\epsilon)} f(i+\epsilon, j+\epsilon, k+\epsilon) & \text{if } i+j+k \mod 2 = 0 \end{cases}$$

The limit can be easily computed for all $i, j, k \in \mathbb{N}^*$ but there are no closed form expression for the limit valid for all i, j, k. The limit depends in fact on the order of i, j, k. We choose such a complicated formula because of its generality. It holds in all cases and thus allows to speed up the proof, avoiding to do 3 times the same, and shows effectively the symmetry between the indexes. With these formulas, it will be easy to prove Theorem 9 because the cases $\partial_\alpha S_{i,j,k} = 0$ correspond to singular values of the Γ functions in the denominator.

Proof. **The case $i + j + k$ mod $2 = 1$.** We begin by looking at f for $i + j + k$ mod $2 = 1$. This is the easy case, because when we replace ϵ by 0 in $f(i+\epsilon, j+\epsilon, k+\epsilon)$, the expression $f(i,j,k) \quad i,j,k \in \mathbb{N}^*$ is still meaningful if we assume $\Gamma(-n) = \infty, \ n \in \mathbb{N}$. Indeed, there can be at most one term of this kind and always in the denominator. The corresponding value of $S_{i,j,k}$ will be 0. We then check that this expression formally satisfies the recurrences (2.13). We now need to look at boundary values. First of all, we remark that one term in the relations (2.13) disappears for some specific i, j, k. We want to prove that a solution for $i + j + k$ mod $2 = 1$ to the recurrences (2.13) is uniquely determined by its values on the axis $i = j = k$.

Let us look first at the case $i + j + k = 3$ mod 6. Using the first relation of (2.13) (and its permutations), we can express $S_{i,j,k}$ in function of $S_{2n+1,2n+1,2n+1}$ with $i+j+k = 6n+3$ (such a $n \in \mathbb{N}$ always exists) if

$$3+i+j-k > 0 \qquad 3+i-j+k > 0 \qquad 3-i+j+k > 0 \tag{2.14}$$

because in this case both terms appear in this first relation (and all its permutations). Indeed, we proceed step by step going closer to the central axis $i = j = k$ for which the condition (2.14) is clearly satisfied. If the condition (2.14) is not satisfied, we can also go closer to the central axis but we meet a singularity when one of the 3 quantities of (2.14) vanishes and this requires that $S_{i,j,k} = 0$ in this case. So if condition (2.14) is not satisfied, we get $S_{i,j,k} = 0$, and we can check it is compatible with our formula. Let us look now at the cases $i + j + k = 1, 5 \bmod 6$. We cannot go exactly to the axis because of that, but we can always go closely, for example to the cases $S_{2n+1,2n+1,2n-1}, S_{2n+1,2n+1,2n+3}$ (which always satisfy condition (2.14)). It is now just necessary to use the second relation of (2.13).

$$(n+1)^2(6n+5)S_{2n+1,2n+1,2n+1} - 8(2n-1)^2(n+2)n(3n+1)S_{2n+1,2n+1,2n-1} = 0$$
$$(n+2)^2(2n-1)(6n+7)S_{2n+1,2n+1,2n+3} - 8(2n+1)^2(n+1)^2(2n+3)(3n+2)S_{2n+1,2n+1,2n+1} = 0$$

The coefficients of these relations are never zero for $n \in \mathbb{N}^*$, so we can express $S_{2n+1,2n+1,2n-1}$, $S_{2n+1,2n+1,2n+3}$ in function of $S_{2n+1,2n+1,2n+1}$.

To conclude, we just need to check our formula on the axis $i = j = k$. Using again holonomic computation with the computer, we get the following relation

$$(n+3)^3(3n+4)(3n+8)S_{n+2,n+2,n+2} - 64n^3(n+1)^4(n+2)^2(3n+1)(3n+5)S_{n,n,n} = 0$$

Our expression satisfies it and so it is only necessary to check the initial value. We have

$$\operatorname*{Res}_{t=\infty} (\epsilon_1^{-1}Q_1(t) + \alpha P_1)^3(t^2-1)^2 = 8\alpha^2/5$$

and this fits our formula.

The case $i+j+k$ mod $2 = 0$. We now look at the function f for $i+j+k \bmod 2 = 0$. This time, if we replace formally $\epsilon = 0$ in $\frac{1}{\Gamma(\epsilon)}f(i+\epsilon, j+\epsilon, k+\epsilon)\alpha$, we find a quotient of Γ functions and in the numerator at most a term of the form $\Gamma(-n)$, $n \in \mathbb{N}$. We can still regularize the formula using the relation $\Gamma(n+1) = n\Gamma(n)$. We get in particular that the limit

$$\lim_{\epsilon \to 0} \frac{1}{\Gamma(\epsilon)} f(i+\epsilon, j+\epsilon, k+\epsilon)$$

is always finite. If there is no term of the form $\Gamma(-n)$, $n \in \mathbb{N}$ in the numerator of f, then the limit is zero. Using invariance by permutation, we will always assume in the following that $k \geq j \geq i$, and so the only possible infinite term in the numerator is the term in $i + j - k$. We get then a zero limit for $k < i + j$, and for $k \geq i + j$, we can regularize the formula. We then check that the formula satisfies the recurrence. Now let us look at the boundary cases. This time we will try to go as far as possible away from the axis $i = j = k$. We want to prove that a solution for $i + j + k \bmod 2 = 0$ to the recurrences (2.13) is uniquely determined by its values on $i = 1, j = 1$.

Using the first relation of (2.13) (and its permutations), we can express $S_{i,j,k}$ in function of $S_{1,\cdot,\cdot}$ if $-2 + i + j - k < 0$ because in that case both terms appear in the first relation of (2.13) (and all its permutations) and because it is satisfied for $i = 1$. If the condition $-2+i+j-k < 0$ is not satisfied, we also can go away from the axis $i = j = k$, but we meet a singularity when $-2+i+j-k = 0$ and this requires that $S_{i,j,k} = 0$ in this case. So if condition $-2+i+j-k < 0$ is not satisfied, we get $S_{i,j,k} = 0$, and we can check it is compatible with our formula. In the case $-1 + j - k < 0$, we can now try to reduce the index j step by step using the relation

$$(2+j-k)(k+1)(1-j+k)S_{1,j,k} - j(-1+j-k)(4-j+k)S_{1,j-1,k+1} = 0$$

We know that $1 - j + k$ never vanishes (it grows at each step), and $2 + j - k = 0$ is not possible because of the parity. So we can always express $S_{1,\cdot,\cdot}$ in function of $S_{1,1,\cdot}$.

To conclude, one just need to compute $S_{1,1,2n+2},\ n \in \mathbb{N}$. We prove the following formula

$$S_{1,1,2n+2} = -8\frac{16^n\Gamma(n+3/2)\Gamma(n-1/2)\Gamma(n+2)\Gamma(n+1)}{\Gamma(n+4)\Gamma(n+5/2)\sqrt{\pi}}$$

using a two terms recurrence and the initial condition $S_{1,1,2} = 16/9\alpha$. We eventually check that it fits our formula. □

To conclude the proof of table A for non zero indexes, we now look at the expression of f. Looking at the formula of $\partial_\alpha S$ for $i+j+k \bmod 2 = 1$, we see that it vanishes exactly when one of the three quantities

$$-i+j+k+3 \qquad -i+j+k+3 \qquad -i+j+k+3$$

is non-positive. This accurately corresponds to the formulas of table A for $i+j+k \bmod 2 = 1$. In the case $i+j+k \bmod 2 = 0$, the quantity

$$\lim_{\epsilon\to 0}\frac{1}{\Gamma(\epsilon)}f(i+\epsilon, j+\epsilon, k+\epsilon)$$

vanishes if and only if all the numbers $-i+j+k, -i+j+k, -i+j+k$ are positive, which is equivalent using the parity condition to

$$-i+j+k \geq 2 \qquad -i+j+k \geq 2 \qquad -i+j+k \geq 2$$

The case of a zero index. We now look at the case with at least one zero index. We can write

$$P_0(t) = \frac{t}{t^2-1} \qquad Q_0(t) = \frac{1}{t^2-1} \qquad Q_i(t) = \epsilon_i P_i(t)\operatorname{arctanh}\left(\frac{1}{t}\right) + \frac{W_i(t)}{t^2-1}$$

(the notation P_0 and Q_0 is here arbitrary because both are rational). We begin with the case where precisely one index is zero. We need to compute the residues

$$\operatorname*{Res}_{t=\infty}\ (\epsilon_i^{-1}Q_i(t)+\alpha P_i)(\epsilon_j^{-1}Q_j(t)+\alpha P_j)(t^2-1)$$

$$\operatorname*{Res}_{t=\infty}\ (\epsilon_i^{-1}Q_i(t)+\alpha P_i)(\epsilon_j^{-1}Q_j(t)+\alpha P_j)t(t^2-1)$$

These are polynomials of degree at most 2 in α but the coefficient in α^2 is always zero because we take the residue of a polynomial at infinity. So one just needs to compute the residue in α. We expand, suppress the polynomial terms, and divide by $\epsilon_i\epsilon_j^{-1} + \epsilon_j\epsilon_i^{-1}$ (which never vanishes) and we get

$$S^1_{i,j} = \operatorname*{Res}_{t=\infty}\ \operatorname{arctanh}\left(\frac{1}{t}\right)P_iP_j(t^2-1) \qquad S^2_{i,j} = \operatorname*{Res}_{t=\infty}\ \operatorname{arctanh}\left(\frac{1}{t}\right)P_iP_jt(t^2-1)$$

One just need to prove that either $S^1_{i,j}$ or $S^2_{i,j}$ is not zero for $i \in \mathbb{N}^*,\ j = i, i+1, i-1$ (the condition on the indexes of table A such that $A = 0$ corresponds here to $-1 \leq i-j \leq 1$). Using the parity on t of the polynomials P_i, we find that only $S^1_{i,i}, S^2_{i,i\pm 1}$ can be non zero. These sequences can be easily computed for finding a recurrence with Mgfun and then a closed form

$$S^1_{i,i} = -2\frac{4^i\Gamma(i+1)^2}{(i+1)(2i+1)i} \qquad S^2_{i,i+1} = -4\frac{4^i\Gamma(i+1)^2}{(2i+1)(2i+3)}$$

These expressions do not vanish. □

2.3.3 Integrability in the cases where $A_{i,j,k} = 1$

Second part: We now prove that if all the non zero terms of second order variational equation correspond only to cases such that $A_{i,j,k} = 1$, then the Galois group is abelian. We use the following lemma

Lemma 8. *We consider*

$$F(t) = \sum_{i=0}^{3} H_i(t) arctanh \left(\frac{1}{t}\right)^i$$

with $H_0, \dots, H_3 \in \mathbb{C}[t]$. If the conditions of Lemma 5 are satisfied, then

- *If $\underset{t=\infty}{Res}\, F(t) = 0$, then $\int F\, dt \in \mathbb{C}\left[t, arctanh\left(\frac{1}{t}\right)\right]$*
- *If $\underset{t=\infty}{Res}\, F(t) \neq 0$, then $\int F\, dt \in \mathbb{C}\left[t, arctanh\left(\frac{1}{t}\right), \ln\left(t^2-1\right)\right]$*

Proof. We proceed using integration by parts. We look at the term of the highest degree and we differentiate $\operatorname{arctanh}(1/t)^3$. Letting

$$J(t) = \int_{-1}^{t} H_3(s) ds$$

we get $J(1) = 0$ using the condition in α^2 of Lemma 5 and performing a series expansion at infinity of $\operatorname{arctanh}(1/t)$. Then $(t^2 - 1)$ divide the polynomial J. After integration by parts, we get a term of the form

$$R(t)\operatorname{arctanh}\left(\frac{1}{t}\right)^2$$

with R a polynomial. Let us try another integration by parts. We get the term

$$\frac{2}{t^2-1}\int \frac{3J(t)}{t^2-1} - H_2(t) dt \operatorname{arctanh}\left(\frac{1}{t}\right)$$

We want this term to be written $Z(t)\operatorname{arctanh}(1/t)$ with Z a polynomial (with a good choice of integration constant). We just need that

$$\int_{-1}^{1} \frac{3J(t)}{t^2-1} - H_2(t) dt = 0$$

Let us look now at the coefficient in α of the residue (2.12). We know it is equal to zero.

$$[\alpha] \underset{t=\infty}{\operatorname{Res}}\, F\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right) + \alpha\right) = \underset{t=\infty}{\operatorname{Res}}\, 3H_3(t)\operatorname{arctanh}\left(\frac{1}{t}\right)^2 + 2H_2(t)\operatorname{arctanh}\left(\frac{1}{t}\right)$$

which gives using an integration by part (we can see the residue as an integration along a small circle around infinity)

$$\underset{t=\infty}{\operatorname{Res}} -\frac{6J(t)}{t^2-1}\operatorname{arctanh}\left(\frac{1}{t}\right) + 2H_2(t)\operatorname{arctanh}\left(\frac{1}{t}\right)$$

Using the Taylor expansion of $\operatorname{arctanh}(1/t)$ at infinity, we get

$$\frac{1}{2}\int_{-1}^{1} -\frac{6J(t)}{t^2-1} + 2H_2(t) dt = 0$$

This is exactly our condition (2.12). So the last remaining integral to compute is of the form

$$\int Z(t)\text{arctanh}\left(\frac{1}{t}\right) dt \in \mathbb{C}\left[t, \text{arctanh}\left(\frac{1}{t}\right), \ln\left(t^2-1\right)\right]$$

which can be proved using an integration by part. Now let us look closer at the possible terms in $\ln\left(t^2-1\right)$. Assume there exists a term $\ln\left(t^2-1\right)$ in $\int F\, dt$. We will have

$$\int F dt = Z_3(t)\text{arctanh}\left(\frac{1}{t}\right)^3 + \cdots + Z_0(t) + r\ln\left(t^2-1\right)$$

with $Z_3,\dots,Z_0$ polynomials and r is a constant because $\ln\left(t^2-1\right)$ does not appear in F. A function in $\mathbb{C}\left[t, \text{arctanh}\left(1/t\right)\right]$ is meromorphic near infinity. Differentiating this expression will give

$$F = g' + \frac{rt}{t^2-1}$$

with g a meromorphic function on a neighbourhood of infinity. Then the residue of F at infinity equals to r. So, if this residue is zero, there will be no $\ln\left(t^2-1\right)$ terms. □

The integrals to compute for the solutions of second order variational method are the following

$$\int (t^2-1)^2 Q_i(t)Q_j(t)Q_k(t)dt \qquad \int (t^2-1)^2 P_i(t)Q_j(t)Q_k(t)dt \tag{2.15}$$

$$\int (t^2-1)^2 P_i(t)P_j(t)Q_k(t)dt \qquad \int (t^2-1)^2 P_i(t)P_j(t)P_k(t)dt \tag{2.16}$$

They all are of the form given by Lemma 5. We already know that the third one and the last one satisfy the condition of Lemma 5, so they all belong to

$$\mathbb{C}\left[t, \text{arctanh}\left(\frac{1}{t}\right), \ln\left(t^2-1\right)\right]$$

For the first one, we use Lemma 7. This proves that the residue is constant for $i, j, k \in \mathbb{N}^*$. For the case $i = 0$, we only need to look at the residues

$$S^1_{i,j} = \underset{t=\infty}{\text{Res}}\ \text{arctanh}\left(\frac{1}{t}\right) P_iP_j(t^2-1) \quad S^2_{i,j} = \underset{t=\infty}{\text{Res}}\ \text{arctanh}\left(\frac{1}{t}\right) P_iP_jt(t^2-1)$$

Performing a Taylor expansion at infinity, we get the formulas

$$S^1_{i,j} = \int_{-1}^{1} P_iP_j(t^2-1)dt \qquad S^2_{i,j} = \int_{-1}^{1} P_iP_jt(t^2-1)dt$$

Moreover we have the property that the family of polynomials P_i is orthogonal for the weight (t^2-1) (due to the orthogonality relations for the Legendre polynomials). So $S^1_{i,j} = 0$ for $i \neq j$ (and we already have computed the case $i = j$ in the previous section). For $S^2_{i,j}$ we use the orthogonal property on P_i and P_jt. If $i \geq j+2$, P_i and P_jt are orthogonal and so $S^2_{i,j} = 0$. By symmetry, it is also the case for $j \geq i+2$. These correspond exactly to the cases $A_{0,i,j} = 1$. Eventually, in the case $i = j = 0$, the first integral of (2.15) can be directly computed using one integration by parts (including the case $i = j = k = 0$).

We now look at the second integral of (2.15). The coefficient in α^2 is automatically 0 because, by expanding the formula, we take the residue of a polynomial. We now want to compute the coefficient in α of the residue. This gives (using a Taylor expansion at infinity)

$$[\alpha] \operatorname*{Res}_{t=\infty} (t^2-1)^2 P_i(t)Q_j(t)Q_k(t) = \frac{1}{2}\epsilon_j\epsilon_k \int_{-1}^{1} P_i(t)P_j(t)P_k(t)(t^2-1)^2 dt$$
$$= [\alpha^2]\epsilon_i^{-1} \operatorname*{Res}_{t=\infty} (t^2-1)^2 Q_i(t)Q_j(t)Q_k(t)$$

which equals to zero because it corresponds to the condition in α^2 for the first integral of (2.15). So the residue condition of Lemma 5 is also satisfied, and then thanks to Lemma 8, it also belongs to

$$\mathbb{C}\left[t, \operatorname{arctanh}\left(\frac{1}{t}\right), \ln\left(t^2-1\right)\right]$$

□

2.3.4 Study of the Galois group in the integrable case

We now prove Theorem 7, analyzing more precisely the Galois group in the case where the integrability conditions of Lemma 5 are satisfied. We will see that in fact the Galois group almost never grows, and the Galois group can be in fact precisely computed thanks to Lemma 8.

Proof of Theorem 7. In the integrable case, the variational equations at order 1 involve the functions P_i, Q_i which are in $\mathbb{C}(t, \operatorname{arctanh}(1/t))$ (after variable change). The only univalued function among the Q_i is the function Q_0, but the eigenvalue 2 is always in the spectrum, and so the Galois group is always at least $\mathbb{C}$. At order 2, using Lemma 5, we already know that the solutions are in $\mathbb{C}\left[t, \operatorname{arctanh}(1/t), \ln\left(t^2-1\right)\right]$ and we know a condition for which the term in $\ln\left(t^2-1\right)$ does not appear. Thanks to Lemma 8, we know that this logarithmic term can appear only if $S_{i,j,k}$ is a non zero constant (independent of α because we assumed that the second order variational equation has a virtually abelian Galois group). Let us prove that

$$S_{i,j,k}|_{\alpha=0} = 0 \;\; \forall\, i,j,k \in \mathbb{N}^* \tag{2.17}$$

We only need to use the recurrence (2.13) for $S_{i,j,k}$. To prove that this sequence is zero, we then only need to prove it vanishes on the boundary. This reduces to the cases $i=j=k=1$ and $i=j=1,\ k=2$ (doing the same as in the proof of Theorem 9). We have $S_{1,1,1} = 8\alpha^2/5$ and $S_{1,1,2} = 16\alpha/9$, and so vanish for $\alpha = 0$. This proves relation (2.17). Let us look now at the case where one of the indexes is zero. We need to study

$$S^1_{i,j} = \operatorname*{Res}_{t=\infty} (t^2-1)Q_i(t)Q_j(t) \qquad S^2_{i,j} = \operatorname*{Res}_{t=\infty} (t^2-1)tQ_i(t)Q_j(t)$$

We also prove they vanish for $\alpha = 0$ using recurrence. If two indexes are zero, then we need to study

$$S^1_i = \operatorname*{Res}_{t=\infty} Q_i(t) \qquad S^2_i = \operatorname*{Res}_{t=\infty} tQ_i(t) \qquad S^3_i = \operatorname*{Res}_{t=\infty} t^2Q_i(t)$$

All these sequences are zero except for $i=1$ for which $S^3_1 = -2/3$. Eventually, in the case where all indexes are zero, we need to compute the following integrals

$$\int \frac{1}{t^2-1}dt \qquad \int \frac{t}{t^2-1}dt \qquad \int \frac{t^2}{t^2-1}dt \qquad \int \frac{t^3}{t^2-1}dt$$

The second and the fourth integral have a term in $\ln\left(t^2-1\right)$. This gives Theorem 7. □

Remark 1. *The computation of the sequence* $S_{i,j,k}|_{\alpha=0}$ *has in fact no sense when* $S_{i,j,k}$ *depend on* α*. Indeed* α *corresponds to the multivaluation of functions* Q*. If* $S_{i,j,k}|_{\alpha=0} = 0$ *and* $S_{i,j,k}$ *depend on* α*, then when we replace* α *by* $\alpha+1$ *(by changing the convention for the definition of functions* Q*), we get* $S_{i,j,k}|_{\alpha=0} \neq 0$*. So in fact this vanishing term only corresponds to a convention taken for the* Q_i*. Still the convention is well chosen, because it allows to study* $S_{i,j,k}|_{\alpha=0}$ *without distinction between integrable cases and non integrable cases (in particular,* $S_{i,j,k}|_{\alpha=0}$ *is almost always zero for all values, and this property allows a much faster proof than in previous sections).*

2.4 Applications

2.4.1 The non diagonalizable case

We now look at the diagonalizability hypothesis of Theorem 6 . We will see it is in fact not a strong condition, because if the first order variational equation has a virtually abelian Galois group, there is only one possibility for which the Hessian matrix can be non-diagonalizable, only for the eigenvalue -1. We do not make a complete analysis of this case at order 2 because it seems not possible to make an efficient reduction in this case to produce a nice criterion, and this case is very rare in practice (still in this case, the Theorem 6 gives integrability constraints on a subsystem, but this constraint is not a priori optimal). So it is probably better adapted to make an analysis case by case directly in applications.

Using our method of analysis for the second order variational equation, let us now reprove the results of Duval and Maciejewski on the non-diagonalizable case.

Corollary 1. *(see a different proof of Duval and Maciejewski in [27]) We consider the equation*

$$\ddot{X}(t) = \frac{1}{\phi(t)^3} A X(t) \tag{2.18}$$

with A *a matrix in Jordan form. Then this equation has a virtually abelian Galois group if and only if*

- $A_{i,i} = \frac{1}{2}(p_i - 1)(p_i + 2)$ *with* $p_i \in \mathbb{N}$
- A *is diagonal except maybe for eigenvalue* -1 *for which the Jordan blocks should be at most of size* 2.

Proof. In the non diagonalizable case, the non homogeneous part of the variational equation corresponds to terms outside the diagonal in the matrix A. For a Jordan block of size 2, the equation (2.18) can be rewritten

$$\ddot{X} = \frac{1}{\phi(t)^3} d_i X + \frac{1}{\phi(t)^3} Y \qquad \ddot{Y} = \frac{1}{\phi(t)^3} d_i Y \tag{2.19}$$

because a Jordan block has the same eigenvalue on the diagonal. For the case with a bigger Jordan block, we would get even stronger conditions because then the equation (2.18) would be a subsystem of equation (2.19). In our case (except for eigenvalue -1), this analysis is not necessary. We compute the solutions and we prove that the following function should be in the Picard-Vessiot field.

$$\int (t^2 - 1)(Q_i(t) + \epsilon_i \alpha P_i)^2 dt$$

With Lemma 5 and Lemma 8, we know it is enough to study the sequence

$$S_i = \operatorname*{Res}_{t=\infty} (t^2 - 1)(Q_i(t) + \epsilon_i \alpha P_i)^2$$

The interesting term is in α, because the coefficient in α^2 is always zero. The Mgfun package give us a recurrence and then a closed form for this residue

$$S_i = -2\frac{4^i i\Gamma(i)^2}{(2i+1)(i+1)}$$

which is never zero for $i \in \mathbb{N}^*$. The case $i = 0$ requires the analysis of a Jordan block of size 3 (higher Jordan block size would still have this system as a subsystem). This gives the equations

$$\begin{aligned} 1/2(t^2-1)\ddot{X}_1 + 2t\dot{X}_1 + X_1 &= 0 \\ 1/2(t^2-1)\ddot{X}_2 + 2t\dot{X}_2 + X_2 &= X_1 \\ 1/2(t^2-1)\ddot{X}_3 + 2t\dot{X}_3 + X_3 &= X_2 \end{aligned} \tag{2.20}$$

Again, we compute the solution and we compute a solution for X_3

$$X_3(t) = (t^2-1)^{-1} \int\int \frac{2t\operatorname{arctanh}\left(\frac{1}{t}\right) + \ln(t^2-1)}{t^2-1} dtdt$$

Thus the following integral belongs to the Picard-Vessiot field

$$\int \frac{2t\operatorname{arctanh}\left(\frac{1}{t}\right) + \ln\left(t^2-1\right)}{t^2-1} dt = \ln(t-1)(2\ln 2 - \ln(t+1)) - 2\operatorname{dilog}\left(\frac{t+1}{2}\right)$$

All the terms are in $\mathbb{C}[t, \operatorname{arctanh}\left(\frac{1}{t}\right), \ln\left(t^2-1\right)]$ except one, the dilogarithmic term

$$\operatorname{dilog}\left(\frac{t+1}{2}\right) = \int \frac{\ln(t+1) - \ln 2}{1-t} dt$$

With the same idea as in Lemma 5, we prove that this term has a non-commutative monodromy because of the following residue at $t = 1$

$$\operatorname*{Res}_{t=1} \frac{\ln(t+1) - \ln 2 + \alpha}{1-t} = -\alpha$$

which depends explicitly on α. To conclude, we notice that the Galois group of equation (2.20) is always connected because we only take recursively integrations (no algebraic functions are involved) and because the Galois group of the first equation of (2.20) is always connected (it is Id). □

2.4.2 A useful corollary

Corollary 2. *Let V be a meromorphic homogeneous potential of degree -1. Let c be a non-degenerate Darboux point of V with multiplier -1. Let $\lambda_1 \dots \lambda_n$ be the eigenvalues of $\nabla^2 V(c)$ with $\lambda_1 = 2$ (the eigenvalue 2 always appears in the spectrum). Assume that $\nabla^2 V(c)$ is diagonalizable and*

$$\begin{aligned} \{\lambda_2, \dots, \lambda_n\} = \{(2k-1)(k+1),\ k \in B\} \\ \textit{with} \quad B \subset \mathbb{N}, \quad \max(B) \leq \max(2\min(B) - 1, 0) \end{aligned} \tag{2.21}$$

Then the variational equation at order 2 near the homothetic orbit associated to c has a virtually abelian Galois group.

Remark 2. *In practice, this corollary tells us that if the eigenvalues of the Hessian matrix have all an even index and are sufficiently close to each other, then the system is always integrable at order 2 without any additional conditions. Moreover, this theorem is in some sense "optimal", because if it is not satisfied, then there will be strong additional integrability constraints (of* *codimension at least* 1*) for integrability. It allows also to have a strong intuition about what will be the easy and the hard cases in proving non-integrability of a particular problem depending on parameters. In practice, this theorem means that the integrability conditions at order* 2 *become stronger and stronger when the dimension* d *and number of Darboux points* k *increase, typically dividing the number of resistant cases (to prove non-integrability) by* 2^{dk}.

Proof. We have the Euler relation due to homogeneity and the Darboux point condition

$$\sum_{i=1}^n q_i \frac{\partial}{\partial q_i} V = -V \qquad \frac{\partial}{\partial q_i} V(c) = -c_i \tag{2.22}$$

By differentiating the Euler relation in q_j, we get

$$\sum_{i=1}^n q_i \frac{\partial}{\partial q_i \partial q_j} V = -2 \frac{\partial}{\partial q_j} V$$

With the Darboux point relation, this implies that c is an eigenvector with eigenvalue 2. Let us note $X_1 = c, X_2, \dots, X_n$ a basis of eigenvectors of $\nabla^2 V(c)$. We will first prove that

$$D^3(V)(c).(X_1, X_a, X_b) = 0 \;\; \forall a \neq b$$

We differentiate the Euler relation (2.22) two times and evaluate on c

$$\sum_{i=1}^n c_i \frac{\partial}{\partial q_i \partial q_j \partial q_k} V(c) = -3 \frac{\partial}{\partial q_j \partial q_k} V(c) \qquad \forall j, k$$

We multiply each line with index j by $(X_a)_j$ and we sum over the j

$$\sum_{j=1}^n \sum_{i=1}^n (X_a)_j c_i \frac{\partial}{\partial q_i \partial q_j \partial q_k} V(c) = -3\lambda_a (X_a)_k \qquad \forall k$$

with λ_a the eigenvalue associated to X_a and using the fact that X_a is an eigenvector of $\nabla^2 V(c)$. We then multiply each line with index k by $(X_b)_k$ and we sum over the k

$$\sum_{k=1}^n \sum_{j=1}^n \sum_{i=1}^n (X_b)_k (X_a)_j c_i \frac{\partial}{\partial q_i \partial q_j \partial q_k} V(c) = -3\lambda_a < X_a | X_b >= 0 \tag{2.23}$$

thanks to orthogonality. This is the expression of $D^3(V)(c).(X_1, X_a, X_b)$. Let us now use Theorem 6. We first remark that all invoked indexes of table A are even. Moreover, for three even indexes i, j, k, if $\max(i, j, k) \leq 2\min(i, j, k) - 2$, then

$$A_{i,i,i}, A_{j,j,j}, A_{k,k,k}, A_{i,j,j}, A_{i,k,k}, A_{j,i,i}, A_{j,k,k}, A_{k,i,i}, A_{k,j,j}, A_{i,j,k} = 1$$

We also have $A_{0,0,0} = 1$. So if the eigenvalues of $\nabla^2 V(c)$ satisfy

$$\mathrm{Sp}(\nabla^2 V(c)) = \{(2k-1)(k+1),\; k \in \tilde{B}\} \text{ with } \tilde{B} \subset \mathbb{N},\; \max(\tilde{B}) \leq \max(2\min(\tilde{B}) - 1, 0)$$

then the system is integrable at order 2. Still, knowing that the eigenvalue 2 always appears in the spectrum of $\nabla^2 V(c)$, this would be useless. But we know that c is always an eigenvector with eigenvalue 2, and that all the possible conditions linked to this eigenvector are of the form $D^3(V)(c).(X_1, X_a, X_b) = 0$. These are automatically satisfied for $a \neq b$. For $a = b$, we have $A_{2,i,i} = 1$ for all $i \neq 1$, so the only possible problem would be if X_a has the eigenvalue 0, but in this particular case, we also get with equation (2.23)

$$\sum_{k=1}^n \sum_{j=1}^n \sum_{i=1}^n (X_a)_k (X_a)_j c_i \frac{\partial}{\partial q_i \partial q_j \partial q_k} V(c) = -3\lambda_a < X_a | X_a >= 0$$

because $\lambda_a = 0$. So the possible integrability conditions involving the eigenvector $X_1 = c$ are always satisfied, and thus we can remove one time the eigenvalue $\lambda_1 = 2$ from the spectrum, and this gives exactly the condition (2.21). □

2.4.3 Some examples

Let us consider the 3 body problem on a line with masses $(1, m, 1)$. The corresponding Hamiltonian is the following

$$H = \frac{p_1^2}{2} + \frac{p_2^2}{2m} + \frac{p_3^2}{2} + \frac{m}{q_1 - q_2} + \frac{m}{q_2 - q_3} + \frac{1}{q_1 - q_3}$$

After the variable change $(p_2, q_2) \longrightarrow (p_2\sqrt{m}, q_2/\sqrt{m})$, the Hamiltonian becomes

$$H = \sum_{i=1}^{3} \frac{p_i^2}{2} + \frac{m}{q_1 - q_2/\sqrt{m}} + \frac{m}{q_2/\sqrt{m} - q_3} + \frac{1}{q_1 - q_3}$$

and so corresponds to a homogeneous potential of degree -1. Let us study this potential using our integrability table.

Proposition 3. *We consider the potential*

$$V = \frac{m}{q_1 - q_2/\sqrt{m}} + \frac{m}{q_2/\sqrt{m} - q_3} + \frac{1}{q_1 - q_3}$$

with $m \in \mathbb{C}$ *and*

$$c = \left(\frac{1}{2}(8m+2)^{1/3}, 0, -\frac{1}{2}(8m+2)^{1/3}\right)$$

Then c *is a Darboux point of* V *with multiplier* -1 *and the identity component of the Galois group of the second order variational equation is abelian if and only if*

$$\frac{8(m+2)}{4m+1} \in \{\tfrac{1}{2}(k-1)(k+2), \quad k \in \mathbb{N}\} \tag{2.24}$$

In this case, the Galois group is then always $\mathbb{C}$*.*

Proof. We first check that c is a Darboux point of V with multiplier -1. We then compute the Hessian matrix of V at c and we find the following spectrum

$$\left\{0, 2, \frac{8(m+2)}{4m+1}\right\}$$

and so the condition at order 1 is exactly the condition 2.24. Looking at the table A of Theorem 6, we have $A_{k,k,k} = 1$ if k even and $A_{k,k,k} = 0$ if k odd. As proved in Corollary 2, the eigenvalue 2 can be discarded because anyway the corresponding integrability conditions will be satisfied. Let us look at integrability conditions of Theorem 6. The eigenvalue 0 has for eigenvector $X_1 = (1, \sqrt{m}, 1)$, and the potential is invariant by translation along this vector. Thus all corresponding derivatives of V will vanish. The only possible integrability constraint can come from the cases $A_{k,k,k} = 0$ corresponding to k odd. We find that

$$D^3V(c).(X_3, X_3, X_3) = 0 \quad \text{with} \quad X_3 = \left(-\sqrt{m}, 2, -\sqrt{m}\right)$$

and so the constraint is always satisfied. Using moreover Theorem 7, we find that the Galois group is always $\mathbb{C}$. □

This case is an unfortunate case because variational equations of order 2 give nothing more than order 1, and order 3 would be necessary to conclude using solely this Darboux point. The problem is still tractable if one uses the other Darboux points. Still, some "resistant" potentials require in all cases higher variational equations.

Proposition 4. *Let us consider the potential*

$$V(r,\theta) = r^{-1}\left((1+a) - 2ae^{i\theta} + ae^{2i\theta}\right) \quad a \in \mathbb{C}$$

where r,θ correspond to polar coordinates in the plane. If V is meromorphically integrable, then

$$-2a-1 \in \{(2k-1)(k+1),\ \ k \in \mathbb{N}\}$$

and in this case, the Galois group of the second order variational equation near the homothetic orbit associated to the unique Darboux point is always $\mathbb{C}$.

Proof. Let us search first the Darboux points. After computation, we find only one Darboux point which is $c = (1,0)$ and has multiplier -1. The spectrum of the Hessian matrix at c is given by $\{2, -2a-1\}$. The integrability condition of order 1 gives that

$$-2a-1 = \frac{1}{2}(k-1)(k+2) \quad k \in \mathbb{N}$$

The only condition of order 2 can come from the eigenvalue $-2a-1$, whose eigenvector is $X = (0,1)$. Looking at the table A of Theorem 6, we have $A_{k,k,k} = 1$ if k even and $A_{k,k,k} = 0$ if k odd. For k even we have no conditions at all, and for k odd, we get the condition

$$D^3(V)(c).(X,X,X) = -6ia = 0$$

For k odd, the only possible case is then $a = 0$, but this implies that $-2a-1 = -1$ which corresponds to $k = 0$ (even). So the case k odd is never possible. Eventually we compute the Galois group, and we find that for k even it is always $\mathbb{C}$, including the case $k = 0$ because the condition of Theorem 7 is satisfied (we have $D^3(V)(c).(X,X,X) = 0$). □

Remark 3. *Remark that in this second example, there is a square root in the potential. As explained in chapter 1 (Combot [21]), the potential V is in fact well defined on the complex manifold $\mathcal{S} = \{(q_1,q_2,r),\ \ r^2 = q_1^2+q_2^2,\ q \neq 0\}$ (instead of $\mathbb{C}^n$ as in the whole chapter). On this complex manifold $\mathcal{S}$, V is meromorphic (and even rational), and the corresponding Hamiltonian is holomorphic on $\mathbb{C}^2 \times \mathcal{S}$. We can apply the most general version of Morales-Ramis-Simó Theorem 4, using the orbit (c is a Darboux point)*

$$\Gamma = \{p = \dot{\phi}.c,\ q = \phi.c,\ r = \mid c \mid \phi,\ \tfrac{1}{2}\dot{\phi} = 1/\phi + 1,\ \phi \neq 0\} \subset \mathbb{C}^2 \times \mathcal{S},$$

the Hamiltonian H being holomorphic on a neighbourhood of Γ. Thus Theorems 6, 7 subsequently apply. Remark that, in this second example, the second order integrability condition was useful but not sufficient to solve entirely the problem. This is because infinitely many a remain possible for integrability (as expected, half of these cases were removed, but an infinity still remains). As there is just one Darboux point, using order 2 and then a analysis at order 3 cannot be avoided (which is done in chapter 4).

Chapter 3

Arbitrary order variational equations for generic cases

3.1 Introduction

The problem of finding potential which are integrable in the Liouville sense is a difficult and ancient problem. Liouville found that finding enough first integrals (n for a n-dimensional potential) allow to solve the differential system associated to the potential by quadrature (the potential is then called integrable). The main difficulty is to find these first integrals, as they do not always exists, at least not globally. Almost all rational integrable potential have quite simple first integrals, but one cannot even exclude very high degree rational first integrals. So one of the main problem is to find all integrable potentials, and certify that no others exist.

A Theorem from Morales-Ramis-Simó ([54] Theorem 2) give necessary conditions for integrability with meromorphic first integrals (see also earlier versions of this Theorem in [74, 50, 51]). The differential Galois group of the variational equation near a non-trivial orbit should have an abelian identity component. The main problem is to find this non-trivial orbit, which lead many authors to study homogeneous potentials. Indeed, aside physical interest, such potentials have generically straight line orbits, and thus Morales-Ramis-Simó Theorem can be generically applied. This procedure has lead to many non-integrability proofs and classifications. In particular, Maciejewski-Przybylska found all meromorphically integrable planar polynomial homogeneous potentials of degree $3, 4$ in [44, 46]. In the case of the homogeneity degree -1, many results are linked to the n body problem, which is a classical problem which involve homogeneous potentials of degree -1 [55, 68, 13, 69, 47].

In this chapter, we want to do a similar classification work for the homogeneity degree -1 as Maciejewski-Przybylska for degree $3, 4$ in [44, 46] in the plane. But in contrary these articles dealing only with polynomial potentials, we do not want to put strong restrictions on the potential. The potential $V(q_1, q_2) = (q_1^2 + q_2^2)^{-1/2}$ is "integrable" (two algebraic first integrals), and not even rational. So the only assumption on the potential V will be that V is meromorphic on

$$\mathcal{C} = \{(q_1, q_2, r) \in \mathbb{C}^3, \;\; r^2 = q_1^2 + q_2^2, \;\; r \neq 0\}.$$

We will not succeed in finding all meromorphic integrable potentials of this form, and thus we will add a technical assumption called "eigenvalue bounded" in section 3.2.3, which is generically satisfied. The main theorems of this chapter are the following

Theorem 10. *Let V be a meromorphic homogeneous potential of degree -1 on $\mathcal{C}$ such that $V = r^{-1}U(\theta)$ with U a 2π-periodic meromorphic function and $U(\mathbb{R}) \subset \mathbb{R}$. If V is meromorphically integrable, then*

$$V = \frac{a}{r} \quad a \in \mathbb{R}$$

Theorem 11. *Let V be a meromorphic homogeneous potential of degree -1 on $\mathcal{C}$. We assume there exists $c \in \mathcal{C}$ such that $V'(c) = -c$ and the spectrum of the Hessian matrix of V at c satisfies $\mathrm{Sp}(\nabla^2(V)(c)) \subset \{-1, 0, 2\}$. If V is meromorphically integrable, then V belongs after possibly rotation to one the following families*

$$V = \frac{a}{q_1} + \frac{b}{q_2} \quad a, b \in \mathbb{C}, \qquad V = \frac{a}{r} \quad a \in \mathbb{C} \quad \textit{or}$$

$$V = \frac{a(q_1^2 + q_2^2)}{(q_1 + \epsilon i q_2)^3} + \frac{a}{q_1 + \epsilon i q_2} \quad a \in \mathbb{C}, \; \epsilon = \pm 1 \quad \textit{Hietarinta 1987 [5]}$$

However, not all homogeneous meromorphic potentials on $\mathcal{C}$ admit a Darboux point, and the condition $\mathrm{Sp}(\nabla^2(V)(c)) \subset \{-1, 0, 2\}$ is very restrictive. In section 3.6, we prove that the weaker hypothesis $\mathrm{Sp}(\nabla^2(V)(c)) \subset]-\infty, 27[$ is sufficient to ensure the conclusions of Theorem 11. In particular, this weaker hypothesis does not produces any new meromorphically integrable potentials. We even conjecture that the above potentials are the only integrable ones (conjecture 2). We present an algorithm able to prove (or disprove) this conjecture, and from this follows a method to prove that no other integrable potentials exists with $\mathrm{Sp}(\nabla^2(V)(c)) \subset]-\infty, M[$

for any fixed $M > 2$. Following this conjecture, we also give all possible abelian Galois groups that could appear in higher variational equation, depending on the derivatives of the potential V at c. On practical problems (potentials with finitely many parameters), such an assumption can be automatically checked thanks to real algebraic geometry algorithms like RAGlib [26]. Although generic on such parametrized potentials, this assumption is not always satisfied (see Combot-Koutschan [22]).

In section 3.2, we present several properties of homogeneous potentials on $\mathcal{C}$, including the "eigenvalue bounded" property. In section 3.3, we prove that a real homogeneous potential in q_1, q_2, c has always straight line orbits, and if meromorphically integrable, satisfies the hypotheses of Theorem 11. Thus Theorem 11 implies Theorem 10. The section 3.4 presents some properties of higher variational equations, and in particular a notion of non-degeneracy. We prove in particular that this property is often satisfied by higher variational equation, and imply a uniqueness Theorem 12. The section 3.5 deal with a special case where the non-degeneracy property is not satisfied, and thus requiring a clever analysis of higher variational equations. The section 3.6 deals with eigenvalues other than $-1, 0, 2$. We present and partially prove conjecture 2, and an algorithm able to analyze higher variational equations up to order 7 with parameters. The last section deals with degenerate cases, where few integrability conditions exist. We eventually give several homogeneous potentials on $\mathcal{C}$ for which we were not able to conclude, because of the lack of straight line orbits or too degenerate ones.

3.2 Generalities

3.2.1 Homogeneous potentials

We will consider from now a Hamiltonian system given by

$$H(p_1, p_2, q_1, q_2, r) = \frac{1}{2}(p_1^2 + p_2^2) - V(q_1, q_2, r) \quad \text{with} \quad (q_1, q_2, r) \in \mathcal{C}$$

where $\mathcal{C}$ is an algebraic manifold

$$\mathcal{C} = \{(q_1, q_2, r) \in \mathbb{C}^3, \;\; r^2 = q_1^2 + q_2^2, \;\; r \neq 0\}$$

The Hamiltonian H is associated to a dynamical system X_H on $\mathbb{C}^2 \times \mathcal{C}$ given by the following equations

$$\begin{gathered} \dot{q_1} = p_1 \qquad \dot{q_2} = p_2 \qquad \dot{r} = r^{-1}(p_1 q_1 + p_2 q_2) \\ \dot{p_1} = \frac{\partial}{\partial q_1} H + r^{-1} q_1 \frac{\partial}{\partial r} H \qquad \dot{p_1} = \frac{\partial}{\partial q_2} H + r^{-1} q_2 \frac{\partial}{\partial r} H \end{gathered} \tag{3.1}$$

This is a 2 degrees of freedom Hamiltonian system, and the Hamiltonian H is meromorphic on $\mathbb{C}^2 \times \mathcal{C}$. The symplectic structure on $\mathcal{C}$ is defined by the derivations in p, q. The potential V will now be assumed to be meromorphic on $\mathcal{C}$ and homogeneous of degree -1

Definition 5. *We say that a meromorphic potential on $\mathcal{C}$ is homogeneous of degree -1 if for all $(q_1, q_2, r) \in \mathcal{C}$ and $\alpha \in \mathbb{C}^*$ we have*

$$V(\alpha q_1, \alpha q_2, \alpha r) = \alpha^{-1} V(q_1, q_2, r)$$

This type of potentials contains some useful algebraic potentials as $(q_1^2 + q_2^2)^{-1/2}$. The construction of this Hamiltonian system is also done in [21] to define algebraic potentials. On $\mathcal{C}$, we can now properly define polar coordinates

Proposition 5. *Let V be a homogeneous potential of degree -1 meromorphic on $\mathcal{C}$. There exists U, 2π-periodic and meromorphic in θ, such that for all $(q_1,q_2,r)\in\mathcal{C}$, we have*

$$V(q_1,q_2,r)=r^{-1}U(\theta)\quad \text{with}\quad r\cos\theta=q_1,\;\; r\sin\theta=q_2 \tag{3.2}$$

Conversely, if V can be written as (3.2), then V is a homogeneous potential of degree -1 meromorphic on $\mathcal{C}$.

Proof. Assume that

$$V(q_1,q_2,r)=r^{-1}U(\theta)\quad \text{with}\quad r\cos\theta=q_1,\;\; r\sin\theta=q_2,\;\; r^2=q_1^2+q_2^2$$

with U meromorphic 2π-periodic. As U is meromorphic 2π-periodic, it can be written $U(\theta)=f(\exp i\theta)$ with f meromorphic. We also have

$$\exp i\theta=\frac{q_1+iq_2}{r}$$

and thus we obtain

$$V(q_1,q_2,r)=r^{-1}f\left(\frac{q_1+iq_2}{r}\right)$$

As f is meromorphic, the function V is meromorphic in its 3 variables for any $(q_1,q_2,r)\in\mathbb{C}^2\times\mathbb{C}^*$. Conversely, using homogeneity, we have

$$V(q_1,q_2,r)=r^{-1}V(\cos\theta,\sin\theta,1)\qquad \forall (q_1,q_2,r)\in\mathcal{C}$$

using $r\cos\theta=q_1,\; r\sin\theta=q_2$. The function $V(\cos\theta,\sin\theta,1)$ is meromorphic 2π-periodic. □

Thanks to Proposition 5, it is equivalent to study the potentials of the form $V=r^{-1}U(\theta)$ with U a 2π-periodic meromorphic function. In polar coordinates, a rotation of a potential $V=r^{-1}U(\theta)$ of an angle $\theta_0\in\mathbb{C}$ is simply the potential $V=r^{-1}U(\theta+\theta_0)$. A motivation to consider homogeneous potentials on $\mathcal{C}$ instead of $\mathbb{C}^2$ is that we want to include the rotation-invariant potential $V=1/r$, which is always integrable in the sense of the following definition

Definition 6. *Let V be a meromorphic homogeneous potential of degree -1 on $\mathcal{C}$. We will say that V is meromorphically integrable if there exists a first integral I of X_H (given by equation* (3.1)*) meromorphic on $\mathbb{C}^2\times\mathcal{C}$ and functionally independent with H.*

To study the meromorphic integrability of such potentials, the main tool we will use is the Morales-Ramis-Simó Theorem

Theorem 12. *(Morales-Ramis-Simó [54] Theorem 2.) Let us consider a symplectic analytical complex manifold M of dimension $2n$, with the Poisson bracket defined by the symplectic form, H a Hamiltonian analytic on M and $\Gamma\subset M$ a particular (not a point) orbit. If H admits a complete system of first integrals in involution, functionally independent and meromorphic on a neigbourhood of Γ, then the identity component of the Galois group of variational equations is abelian at any order.*

A definition of variational equations can be found in [54] page 860. For meromorphic potentials on an algebraic manifold, we have moreover the following Theorem

Theorem 13. *(Combot [21] Theorem 2. page 3) Let V be a meromorphic potential on an open set $U\subset\mathcal{C}$ and $\Gamma\subset\mathbb{C}^2\times U$ a non-stationary orbit of V. Assume $\Gamma\not\subset\mathbb{C}^2\times\Sigma(V)$ where $\Sigma(V)$ is the set of singularities of V. If there is a first integral meromorphic on $\mathbb{C}^2\times(U\setminus\Sigma(V))$ of V functionally independent with the Hamiltonian H over an open neighbourhood of Γ, then the identity component of Galois group of the variational equation near Γ is abelian over the base field of meromorphic functions on $\Gamma\setminus(\mathbb{C}^2\times\Sigma(V))$.*

In the original statement of [21], the set $\Sigma(V)$ not only contains singular points of V but also points where V is not C^∞ (with respect to derivations in q). As the algebraic extension $r=(q_1^2+q_2^2)^{1/2}$ is always differentiable on $\mathcal{C}$ (because $r\neq 0$ on $\mathcal{C}$), so is the potential V outside its singularities.

3.2.2 Darboux points

In Theorem 13, one of the ingredients is the orbit Γ. To find such an orbit of our Hamiltonian system, we will use Darboux points

Definition 7. *We say that $c \in \mathcal{C}$ is a* Darboux point *of V a meromorphic homogeneous potential of degree -1 on $\mathcal{C}$ if*

$$\frac{\partial}{\partial q_1} V(c) = \alpha c_1 \qquad \frac{\partial}{\partial q_2} V(c) = \alpha c_2 \tag{3.3}$$

The number $\alpha \in \mathbb{C}$ is called the multiplier *associated to c. We say that c is non-degenerate (or improper in [46]) if $\alpha \neq 0$.*

In the non-integrability setting, these Darboux points are also used in [74, 44, 46, 55, 53] among others. Using homogeneity of V, we can always choose $\alpha = 0, -1$ and so in the following we will always choose the multiplier $\alpha = -1$ for a non-degenerate Darboux point. The most interesting property for us of these Darboux points is that they provide orbits:

Definition 8. *Let $c \in \mathcal{C}$ be a Darboux point of V a meromorphic homogeneous potential of degree -1 on $\mathcal{C}$. A* homothetic orbit of V associated to c *is given by*

$$r(t) = c_3\phi(t) \quad q_i(t) = c_i\phi(t) \quad p_i(t) = c_i\dot{\phi}(t) \quad i = 1, 2$$

(remember that $c_3^2 = c_1^2 + c_2^2$) with ϕ satisfying the following differential equation

$$\frac{1}{2}\dot{\phi}^2 = -\frac{\alpha}{\phi} + E \quad E \in \mathbb{C}$$

This homothetic orbit is used by Morales-Ramis in [53] to build simple integrability conditions using Theorem 12 thanks to the classification of Galois groups of the hypergeometric equation by Kimura [37]. In the case of a homothetic orbit, the first order variational equation is given by

$$\ddot{X} = \frac{1}{\phi(t)^3}\nabla^2 V(c) X$$

where $\nabla^2 V(c)$ is the Hessian matrix of V at c. As the potential is homogeneous, multiplying the value of E does not change the variational equation (up to a change of variable), and so we can always choose $E = 0, 1$. As noted in Appendix A, the case $E = 0$ does not lead to any integrability constraint, and so we will only consider $E = 1$ in the rest of the chapter.

After the variable change $\dot{\phi}/\sqrt{2} \longrightarrow t$ and diagonalization of $\nabla^2 V(c)$ (if possible), the first order variational equation becomes

$$\frac{1}{2}(t^2 - 1)\ddot{X}_i + 2t\dot{X}_i - \lambda_i X_i = 0, \qquad \lambda_i \in \operatorname{Sp}\left(\nabla^2 V(c)\right)$$

Now the integrability condition due to the Galois group of the first order variational equation becomes (according to Morales-Ramis in [53] and [21])

$$\operatorname{Sp}\left(\nabla^2 V(c)\right) \subset \left\{\frac{1}{2}(k-1)(k+2),\ k \in \mathbb{N}\right\} = \{-1, 0, 2, 5, 9, 14, 20, 27, \dots\}$$

Definition 9. *Let $c \in \mathcal{C}$ be a Darboux point of a meromorphic homogeneous potential V of degree -1 on $\mathcal{C}$. We say that V* is integrable at order $k \in \mathbb{N}^*$ at c *if the variational equation of order k of the homothetic orbit associated to c has a Galois group whose identity component is abelian.*

A Darboux point of a meromorphic homogeneous potential V of degree -1 on $\mathcal{C}$ is solution of equation (3.3). Writing V in polar coordinates $V = r^{-1}U(\theta)$, this equation becomes

$$U'(\theta) = 0 \quad \alpha r^3 = -U(\theta)$$

So a non-degenerate Darboux point corresponds to some $\theta \in [0, 2\pi[$ such that

$$U'(\theta) = 0 \quad \text{and} \quad U(\theta) \neq 0$$

Given a 2-dimensional rotation R_{θ_0} of angle θ_0, the symplectic variable change $p = R_{\theta_0} p,\ q = R_{\theta_0} q$ transforms H into the Hamiltonian of the meromorphic homogeneous potential $V(R_{\theta_0} q)$. So meromorphic integrability of the potential $V(R_{\theta_0} q)$ does not depend on θ_0. Looking at equations (3.1), by making a time change we can replace the potential V by γV with $\gamma \in \mathbb{C}^*$ (we will call this transformation a dilatation), as done in [44]. So meromorphic integrability of the potential γV does not depend on $\gamma \in \mathbb{C}^*$.

So up to rotation-dilatation of a potential V, if V has a non-degenerate Darboux point c, we can assume that $c = (1, 0)$ is a non-degenerate Darboux point with multiplier -1, which corresponds in polar coordinates to $U'(\theta) = 0,\ U(\theta) = 1$.

Lemma 9. *Let V be a meromorphic homogeneous potential of degree -1 on $\mathcal{C}$. Assume there exists $c \in \mathcal{C}$ a non-degenerate Darboux point of V. Then after a rotation and dilatation, the potential V has the following properties*

- *The vector $c = (1, 0)$ is a non-degenerate Darboux point of V with multiplier -1.*
- *We have $\mathrm{Sp}(\nabla^2 V(c)) = \{2, \lambda\}$, and the series expansion of V at c is of the form*

$$V(c + q) = 1 - q_1 + q_1^2 + \lambda q_2^2/2 + O(q^3)$$

Proof. As there exists a non-degenerate Darboux point $c \in \mathcal{C}$, we can assume that $c = (1, 0)$ after a rotation (recall that $c_3^2 = c_1^2 + c_2^2 \neq 0$ on $\mathcal{C}$). Multiplying V by a constant, we can assume that $V(c) = 1$ (recall that $V(c) \neq 0$ as c is non degenerate). Using Euler formula, we obtain $\partial_{q_1} V(c) = -V(c)$ and so the multiplier of $c = (1, 0)$ is -1.

Derivating the Euler relation and evaluating it at $(q_1, q_2) = c$, we also have

$$\partial_{q_1} V(c) + \partial_{q_1 q_1} V(c) = -\partial_{q_1} V(c), \qquad \partial_{q_1 q_2} V(c) + \partial_{q_2} V(c) = -\partial_{q_2} V(c)$$

Thus

$$\nabla^2 V(c) c = \begin{pmatrix} \partial_{q_1 q_1} V(c) \\ \partial_{q_1 q_2} V(c) \end{pmatrix} = \begin{pmatrix} -2\partial_{q_1} V(c) \\ -2\partial_{q_2} V(c) \end{pmatrix} = 2c$$

So the eigenvalue 2 always appear in the spectrum. So we can write $\mathrm{Sp}(\nabla^2 V(c)) = \{2, \lambda\}$. The series expansion of V at c follows. $\square$

Example

$$U(\theta) = (1 - \cos(\theta))^n - \frac{n 2^n}{(2k-1)(k+1)+1} - 2^n \qquad n, k \in \mathbb{N}^*$$

The Darboux points of the potential $V = r^{-1}U(\theta)$ correspond to $\theta = 0, \pi$. Computing the eigenvalues at these Darboux points gives respectively the following spectrum of Hessian matrices

$$\{2, -1\} \quad \{2, (2k-1)(k+1)\}$$

These eigenvalues are allowed for meromorphic integrability using Theorem 13 and according to [20], there are no additional integrability conditions at order 2. So this potential is integrable at order 2 near all Darboux points. Moreover, looking at $\theta = 0$, we find that

$$U^{(i)}(0) = 0 \quad i = 1 \ldots 2n - 1$$

This implies that the variational equation of order $2n-2$ of the potential $V = r^{-1}U(\theta)$ is the same as the variational equation of order $2n-2$ of the potential $\tilde{V} = r^{-1}$. This potential $\tilde{V}$ is meromorphically integrable with an additional first integral $p_1q_2 - p_2q_1$ and thus its variational equation of order $2n-2$ has a Galois group whose identity component is abelian. So the potential V is integrable at order $2n-2$ at $\theta = 0$. At $\theta = \pi$, the potential V is probably not integrable at order 3 but it seems quite difficult to prove as the eigenvalue depend on the parameter k which make the higher variational equation very complicated (this problem is still analyzed in [22]).

We could also use the procedure presented in [44, 46] where Maciejewski-Przybylska classify meromorphically homogeneous potentials of degree $3, 4$, but in the case of V this will not work because this method only works for potentials without multiple Darboux points (here the Darboux point corresponding to $\theta = 0$ is multiple for $n \geq 2$). In section 5, we will prove that the potential V is not integrable at order $4n-3$ at $\theta = 0$.

3.2.3 Eigenvalue Bounding

The purpose of this section is to present a new way to analyze integrability of families of planar homogeneous potentials of degree -1. Indeed, the main use of the Morales-Ramis-Simó Theorem (Theorem 12) is to find all meromorphically integrable potentials within a large family. So our main problem is

Problem: Given a set E of meromorphic planar homogeneous potentials of degree -1, find all elements in E which are meromorphically integrable.

We will not be able to solve completely this problem, but we will present an effective way to deal generically with such a question.

Definition 10. *Let us denote*

$$\mathcal{M} = \{V \text{ meromorphic on } \mathcal{C} \text{ homogeneous of degree } -1\}$$

the set of meromorphic homogeneous potentials of degree -1 on $\mathcal{C}$. Let $V \in \mathcal{M}$. We note $d(V)$ the set of Darboux points $c \in \mathcal{C}$ of V with multiplier -1. Given $c \in d(V)$, the spectrum of the Hessian matrix $\nabla^2 V(c)$ always contains the eigenvalue 2 because of the relation $\nabla^2 V(c)c = 2c$ (due to Euler relation for homogeneous functions of degree -1). So we have $\mathrm{Sp}\left(\nabla^2 V(c)\right) = \{2, \lambda\}$ *and we note*

$$\Lambda(c) = \begin{cases} \lambda & \text{if } \lambda \in \mathbb{R} \\ -\infty & \text{otherwise} \end{cases}$$

Let $E subset \mathcal{M}$ be a familly of meromorphic homogeneous potentials of degree -1. Let $V \in \mathcal{M}$. We define the eigenvalue bound *by*

$$\Lambda(E) = \sup_{V \in E,\ d(V) \neq \varnothing} \ \inf_{c \in d(V)} \Lambda(c)$$

Definition 11. *We say that the problem of finding all meromorphically integrable potentials in E is a* bounded eigenvalue problem *if*

$$\Lambda(E) < \infty$$

We have $\Lambda(\mathcal{M}) = \infty$ because of the following family

$$V(r, \theta) = r^{-1}\left((1+a) - 2ae^{i\theta} + ae^{2i\theta}\right), \quad a \in \mathbb{R},$$

for which there exists essentially only one Darboux point corresponding to $\theta = 0$ and the eigenvalues are $\{2, 2a-1\}$. This is typically the special case that Maciejewski-Przybylska detected at the end of [46]. A similar challenge is the analysis of singular perturbations of the potential $1/r$, like the following

$$V = r^{-1}\left(1+\frac{\epsilon}{\cos\theta}\right)$$

which has the same phenomenon when $\epsilon \longrightarrow 0$ (we have $\Lambda = \infty$). One possibility is the holonomic approach of 3-rd order variational equations in [22] which seem to be able to solve the problem.

Still, the "bounded eigenvalue" property is "generic" for rational potentials. This is because there exists a relation (Maciejewski-Pryzbylska [46, 63]) on eigenvalues of Darboux points of the form

$$\sum_{i=1}^{p} \frac{1}{\lambda_i+1} = a \tag{3.4}$$

with p the number of Darboux points and a some constant which only depends on the multiplicity of some roots of V. This relation is valid under some generic assumptions (the Darboux points should be simple, and there are additional restrictions on the multiplicity of the roots $(1, \pm i)$ of V). Still this forbids to have all eigenvalues λ_i simultaneously large, and gives the bound $\Lambda \leq p/a - 1$.

Here we want to propose the following systematic way to prove non integrability. First, we find a majoration of $\Lambda(E)$ of the problem. For real rational potential, several algorithms of real algebraic geometry like RAGlib [26] prove such bounds, or one may use Maciejewski-Przybylska (3.4) relation when it holds. Then we run a program to check non existence of integrable potentials with eigenvalues up to this majoration (this program and associated conjecture are in section "The other eigenvalues" below). Eventually we look inside the family E to check whether there is a potential given by Theorem 11. We do not even have to check Morales-Ramis condition of order 1 given by Theorem 13. This methodology seems to solve many integrability problems. As the bounded eigenvalue property is "generic", we can apply Theorem 11 and the partial proof of conjecture 2 of section 6, used together as a "generic" classification, because it holds for all bounded eigenvalue potentials sets.

3.3 Theorem 11 implies Theorem 10

Lemma 10. *Let V be a meromorphic homogeneous potential on $\mathcal{C}$ of degree -1 such that $V = r^{-1}U(\theta)$ with $U(\mathbb{R}) \subset \mathbb{R}$. Then there exists θ_0 such that*

$$U(\theta_0) \neq 0 \quad U'(\theta_0) = 0 \quad \frac{U''(\theta_0)}{U(\theta_0)} \leq 0$$

Proof. The hypothesis $U(\mathbb{R}) \subset \mathbb{R}$ implies that U is C^∞ on $\mathbb{R}$. The function U is periodic, so there exists a minimum and a maximum for U. Assume first that U is not constant. Then $\max U > \min U$. We have 3 cases

- $\max U \geq \min U \geq 0$. Then we choose θ_0 such that $U(\theta_0) = \max U$
- $\max U \geq 0 \geq \min U$. If $\max U \neq 0$, we choose θ_0 such that $U(\theta_0) = \max U$, otherwise we choose θ_0 such that $U(\theta_0) = \min U$
- $0 \geq \max U \geq \min U$. We choose then θ_0 such that $U(\theta_0) = \min U$

Knowing that $\max U > \min U$, we get $U(\theta_0) \neq 0$. Then in all cases, we have

$$\frac{U''(\theta_0)}{U(\theta_0)} \leq 0$$

Knowing that θ_0 is an extremum, we get

$$U(\theta_0) \neq 0 \quad U'(\theta_0) = 0 \quad \frac{U''(\theta_0)}{U(\theta_0)} \leq 0$$

which gives the theorem. ☐

Let us now prove Theorem 10, assuming Theorem 11.

Proof of Theorem 10. We assume Theorem 11 holds. We let $V = r^{-1}U(\theta)$ in polar coordinates and we use Lemma 10. There exists a θ_0 such that

$$U(\theta_0) \neq 0 \quad U'(\theta_0) = 0 \quad \frac{U''(\theta_0)}{U(\theta_0)} \leq 0$$

We define $c \in \mathcal{C}$ by

$$c_1 = U(\theta_0)\cos\theta_0, \quad c_2 = U(\theta_0)\sin\theta_0 \quad \text{and} \quad c_3 = U(\theta_0)$$

After computation, we find that c satisfies the equation

$$\partial_{q_1} V(c) = -c_1 \qquad \partial_{q_2} V(c) = -c_2$$

So c is a Darboux point of V with multiplier -1. We now compute the eigenvalues of the Hessian $\nabla^2 V(c)$, and we find

$$\mathrm{Sp}(\nabla^2 V(c)) = \left\{2, \frac{U''(\theta_0)}{U(\theta_0)} - 1\right\}$$

If V is meromorphically integrable than, thanks to Theorem 13, the eigenvalues at Darboux points should belong to

$$\left\{\frac{(p-1)(p+2)}{2},\ p \in \mathbb{N}\right\} = \{-1, 0, 2, 5, 9, 14, \dots\}$$

But here we have moreover that

$$\frac{U''(\theta_0)}{U(\theta_0)} \leq 0$$

So this implies that in fact

$$\mathrm{Sp}(\nabla^2 V(c)) = \{2, -1\}$$

This case is in the hypotheses of Theorem 11. Among the 3 possibles families, only the second one has the eigenvalue -1. Using the fact that U should be a real function (and non zero), this implies that $V = ar^{-1},\ a \in \mathbb{R}^*$. ☐

Lemma 10 has then proved the following

$$\Lambda\left(\left\{V = r^{-1}U(\theta) \ \ U \neq 0 \text{ meromorphic } 2\pi\text{-periodic } U(\mathbb{R}) \subset \mathbb{R}\right\}\right) = -1$$

So this set satisfies our "bounded eigenvalue" property. As we see, it is not necessary to have a finite dimensional family of homogeneous potentials to have the "bounded eigenvalue" property, and so quite large families could be analyzed.

3.4 Non degeneracy of higher variational equations

3.4.1 First order variational equations

For now, we proved that after reduction, a meromorphic homogeneous potential on $\mathcal{C}$ possessing a non-degenerate Darboux point, can be assumed to have to following properties

- The point $c = (1, 0)$ is a Darboux point of V with multiplier -1 and $V(c) = 1$.
- The spectrum of the Hessian matrix of V at c is of the form $\text{Sp}(\nabla^2 V(c)) = \{2, \lambda\}$.
- The first order variational equation near a homothetic orbit with energy $E = 1$ is given by (after variable change $\dot{\phi}/\sqrt{2} \longrightarrow t$)

$$\frac{1}{2}(t^2-1)\ddot{y} + 2t\dot{y} - \frac{1}{2}(n-1)(n+2)y = 0 \qquad n \in \mathbb{N} \tag{3.5}$$

- If the first order variational equation has a Galois group whose identity component is abelian, then
$$\lambda \in \{\tfrac{1}{2}(n-1)(n+2),\ n \in \mathbb{N}\}$$

We now recall some properties of the solutions of the first order variational equation 3.5. A basis of solutions is given by (P_n, Q_n) where P_n is polynomial in t of degree $n-1$ and the functions Q_n are

$$Q_n(t) = P_n(t) \int \frac{1}{(t^2-1)^2 P_n(t)^2} dt$$

The functions Q_n are multivalued except for $n = 0$ which will be a special case. Indeed, the Galois group of (3.5) in this case is $\{Id\}$ instead of $\mathbb{C}$.

The polynomials P_n can be computed using the "Rodrigues" type formula

$$P_n(t) = \frac{1}{t^2-1} \frac{\partial^{n-1}}{\partial t^{n-1}} (t^2-1)^n$$

which gives a normalization for the leading term of P_n that we will choose now and the functions Q_n can be written as

$$Q_n(t) = \epsilon_n P_n(t) \operatorname{arctanh}\left(\frac{1}{t}\right) + \frac{W_n(t)}{t^2-1}$$

with W_n being polynomials, and ϵ_n a real sequence given by

$$\epsilon_n = \frac{4^{-n} n(n+1)}{n!^2}$$

Lemma 11. *Let $F \in \mathbb{C}(z_1)[z_2]$ and*

$$f(t) = F\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right)\right) \in \mathbb{C}(t)\left[\operatorname{arctanh}\left(\frac{1}{t}\right)\right]$$

We consider the following differential field extension and its differential Galois group

$$K = \mathbb{C}\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right), \int f dt\right), \qquad G = Gal_{diff}(K/\mathbb{C}(t))$$

If G is abelian, then

$$\frac{\partial}{\partial \alpha} \underset{t=\infty}{Res}\ F\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right) + \alpha\right) = 0 \quad \forall \alpha \in \mathbb{C}$$

where Res correspond to the residue.

Proof. We first remark that the Zariski closure of the monodromy group of f in the complex plane $\mathbb{C}\setminus\{-1,1\}$ is exactly the Galois group G because f satisfies a linear differential equation whose singularities are regular. We now consider two paths, the "eight" path σ_1 around the singularities -1 and 1, and the path σ_2 around infinity. At infinity, the function $F\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right)+\alpha\right)$ will have a series expansion of the kind

$$\int F\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right)+\alpha\right)dt = \sum_{n=n_0}^{\infty} a_n(\alpha)t^n + r(\alpha)\ln\ t$$

because the function $\operatorname{arctanh}\left(\frac{1}{t}\right)$ has a regular point at infinity. Let us now consider the monodromy commutator

$$\sigma = \sigma_2^{-1}\sigma_1^{-\frac{\beta}{2i\pi}}\sigma_2\sigma_1^{\frac{\beta}{2i\pi}} \quad \text{with} \quad \beta\in 2i\pi\mathbb{Z}$$

Computing the monodromy, we obtain $\sigma_1^{\frac{\beta}{2i\pi}}(f) = F\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right)+\beta\right)$ and $\sigma_2(\ln t) = \ln t + 2i\pi$. We deduce that

$$\sigma(f) = f + r(\beta) - r(0)$$

This $r(\beta)$ corresponds to the residue of $F\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right)+\beta\right)$ at infinity. If G is abelian, then the monodromy is commutative, and then the commutator σ should act trivially on f. This is the case only if $r(\beta) = r(0)\ \ \forall\beta\in 2i\pi\mathbb{Z}$. The function r is a polynomial in β, so $r(\beta) - r(0),\ \ \forall\beta\in\mathbb{C}$. This gives us the formula of the lemma

$$\frac{\partial}{\partial\alpha}\operatorname*{Res}_{t=\infty}\ F\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right)+\alpha\right) = 0 \quad \forall\alpha\in\mathbb{C}$$

□

3.4.2 Higher order variational equations

We now follow the definition of higher variational equations given by Morales-Ramis-Simó [54] page 860. Using their notation, the variational equations can be written

$$\dot{\varphi}_t^{(1)} = X_H^{(1)}\varphi_t^{(1)}$$

$$\dot{\varphi}_t^{(2)} = X_H^{(1)}\varphi_t^{(2)} + X_H^{(2)}(\varphi_t^{(1)})^2$$

$$\dot{\varphi}_t^{(3)} = X_H^{(1)}\varphi_t^{(3)} + 2X_H^{(2)}(\varphi_t^{(1)},\varphi_t^{(2)}) + X_H^{(3)}(\varphi_t^{(1)})^3$$

and they give a formula for any order k. In particular, at any order k, the last equation has always the following structure. There is a homogeneous part $\dot{\varphi}_t^{(k)} = X_H^{(1)}\varphi_t^{(k)}$, and non homogeneous terms involving functions already computed when solving lower order variational equations. So this last equation can be considered as a non homogeneous linear equation.

We still assume that we are in the homogeneous potential case, with a non degenerated Darboux point at $c=(1,0)$ with multiplier -1. The X_H is the Hamiltonian field, and we may write $\varphi_t^{(k)} = (\dot{X}_1,\dot{X}_2,X_1,X_2)$. The X_1 correspond to a perturbation tangential to the homothetic orbit, and X_2 normal to this orbit. We see also that this variational equation is not linear. But for example at order 3, instead of considering non linear terms like $(\varphi_t^{(1)})^3$, we replace it by solutions of the symmetric power of the equation satisfied by $\varphi_t^{(1)}$ (for this term, this gives the third symmetric power of the first order variational equation).

Computing variational equations up to order k will produce monomials in the components of vectors $\varphi_t^{(1)},\dots,\varphi_t^{(k)}$. Equation (13) of [54] can be rewritten

$$\dot{\varphi}_t^{(k)} = \sum_{j=1}^{k}\sum \frac{j!}{m_1!\dots m_s!}X_H^{(j)}((\varphi_t^{(i_1)})^{m_1},\dots,(\varphi_t^{(i_s)})^{m_s})$$

For each fixed j, the inner sum is a sum monomials of the form

$$(\varphi_t^{(1)})_{w_1}^{j_1} \dots (\varphi_t^{(k)})_{w_k}^{j_k} \tag{3.6}$$

where w indicates the component of vectors φ_t. Instead of computing $\varphi_t^{(i)}$, we compute directly these monomials. We note y_{n_1,n_2,n_3,n_4} the sum over all monomials (3.6) having exactly n_1 terms with $w = 1$, n_2 terms with $w = 2$, etc. Due to symmetries of higher variational equations, considering these y_{n_1,n_2,n_3,n_4} are sufficient to analyze the variational equation (meaning that the derivatives of y only involve y). This process has also linearized the variational equation as y_{n_1,n_2,n_3,n_4} correspond to the monomials in the sum themselves. Building linear differential equations for the y_{n_1,n_2,n_3,n_4} necessitates by the way to compute the symmetric product of differential systems (as done [4]), as we need to build linear differential equation satisfied by monomials of the form (3.6). At order k, the variational equation now writes

$$\begin{pmatrix} \ddot{y}_{0,0,1,0} \\ \ddot{y}_{0,0,0,1} \end{pmatrix} = \frac{1}{\phi^3} \begin{pmatrix} 2 & 0 \\ 0 & \lambda \end{pmatrix} \begin{pmatrix} y_{0,0,1,0} \\ y_{0,0,0,1} \end{pmatrix} + \begin{pmatrix} \sum_{i=2}^{k} \frac{1}{\phi^{i+2}} \sum_{j=0}^{i} \frac{d_{i,j}}{(i-j)!j!} y_{0,0,i-j,j} \\ \sum_{i=2}^{k} \frac{1}{\phi^{i+2}} \sum_{j=0}^{i} \frac{d_{i,j+1}}{(i-j)!j!} y_{0,0,i-j,j} \end{pmatrix} \tag{3.7}$$

where $y_{i,0,j,0}(t)$ satisfy differential equations corresponding to lower order variational equations. The coefficients $d_{i,j}$ are given by

$$d_{i,j} = \frac{\partial^{i+1}}{\partial q_1^{i-j+1} \partial q_2^j} V(c)$$

A visual process to build these differential systems is to see y_{n_1,n_2,n_3,n_4} as $\dot{X}_1^{\,n_1} \dot{X}_2^{\,n_2} X_1^{n_3} X_2^{n_4}$. We differentiate this expression and simplify it using the relation

$$\ddot{X} = \frac{1}{\phi^3} \begin{pmatrix} 2 & 0 \\ 0 & \lambda \end{pmatrix} X + \begin{pmatrix} \sum_{i=2}^{k} \frac{1}{\phi^{i+2}} \sum_{j=0}^{i} \frac{d_{i,j}}{(i-j)!j!} X_1^{i-j} X_2^j \\ \sum_{i=2}^{k} \frac{1}{\phi^{i+2}} \sum_{j=0}^{i} \frac{d_{i,j+1}}{(i-j)!j!} X_1^{i-j} X_2^j \end{pmatrix} \tag{3.8}$$

We suppress terms degree $> k$ that could appear, and then we formally replace back the $\dot{X}_1^{\,n_1} \dot{X}_2^{\,n_2} X_1^{n_3} X_2^{n_4}$ by y_{n_1,n_2,n_3,n_4}.

Remark 4. *Remark now that using the Euler relation for homogeneous function*

$$q_1 \partial_{q_1} V + q_2 \partial_{q_2} V = -V$$

and derivating it in q_1 or q_2 enough at $(q_1, q_2) = (1, 0)$, we obtain the relations

$$\partial_{q_1^i q_2^j} V + \partial_{q_1^i q_2^j} V + \partial_{q_1^{i+1} q_2^j} V = -\partial_{q_1^i q_2^j} V, \quad i \geq 1, j \geq 0$$

This gives all derivatives $d_{k,j}$ of V of order $k+1$ in function of lower order ones except $d_{k,k+1}$.

By construction, the differential equations for the y_{n_1,n_2,n_3,n_4} have special structure. In particular, the expression of $\dot{y}_{n_1,n_2,n_3,n_4}$ only involve terms with higher or equal sum of indexes. Thus, in particular, the differential equation for $y_{n_1,n_2,n_3,n_4},\ n_1 + n_2 + n_3 + n_4 = k$ is linear homogeneous and correspond to the k-th symmetric power of the first order variational equation. So the $y_{n_1,n_2,n_3,n_4},\ n_1 + n_2 + n_3 + n_4 = k$ are linear combinations of product of degree k of solutions of the first order variational equation, which will be in our case after a variable change products of degree k functions P, Q.

Let us now look at equation (3.7). This is a non homogeneous linear equation, so once we have found the expression of the non homogeneous term, we can solve it using variation of parameters. Remark also that the highest order derivatives $d_{k,k+1}$ of V at c only appears in this equation and not the lower order ones. We write the solution of the second equation of (3.7)

$$y(t) = y_{hom}(t) + y_{part_1}(t) + d_{k,k+1} y_{part_2}(t) \tag{3.9}$$

isolating the term in $d_{k,k+1}$. The part $y_{hom}(t)$ is a solution of the homogeneous part, the solution $y_{part_1}(t)$ is a particular solution of equation (3.7) without the term in $d_{k,k+1}$ and $y_{part_2}(t)$ is a particular solution of equation (3.7) where all non homogeneous terms are removed except the one in in $d_{k,k+1}$. Let us apply a monodromy commutator

$$\sigma_\alpha = \sigma_2^{-1} \sigma_1^{-\frac{\alpha}{2i\pi}} \sigma_2 \sigma_1^{\frac{\alpha}{2i\pi}}$$

(with the same notation as in the proof of Lemma 11). This gives

$$\sigma_\alpha(y) = \sigma_\alpha(y_{hom}) + \sigma_\alpha(y_{part_1}) + d_{k,k+1}\sigma_\alpha(y_{part_2})$$

Now let us look at y_{part_2}. If $\lambda = \frac{1}{2}(g-1)(g+2),\ g \in \mathbb{N}$, it can be computed and one solution is

$$y_{part_2}(t) = \int (t^2-1)^k Q_g^{k+1} dt$$

We now apply lemma 11 which says that the monodromy element σ add to such function the constant

$$G(\alpha) = \operatorname*{Res}_{t=\infty} \ (t^2-1)^k (Q_g + \epsilon_g \alpha P_g)^{k+1} - \operatorname*{Res}_{t=\infty} \ (t^2-1)^k Q_g^{k+1}$$

The equation $\sigma_\alpha(y) = y$ becomes

$$\begin{aligned} 0 = \sigma_\alpha(y) - y = \\ \sigma_\alpha(y_{hom}) + \sigma_\alpha(y_{part_1}) + d_{k,k+1}\sigma_\alpha(y_{part_2}) - y_{hom} - y_{part_1} - d_{k,k+1} y_{part_2} = \\ \sigma_\alpha(y_{hom}) + \sigma_\alpha(y_{part_1}) - y_{hom} - y_{part_1} + d_{k,k+1}(\sigma_\alpha(y_{part_2}) - y_{part_2}) = \\ \sigma_\alpha(y_{hom}) + \sigma_\alpha(y_{part_1}) - y_{hom} - y_{part_1} + d_{k,k+1} G(\alpha) \end{aligned}$$

If the function $G(\alpha)$ is not zero (which is equivalent to G being non-constant), then this system of equations in for $\alpha \in 2i\pi\mathbb{Z}$ has at most one solution in $d_{k,k+1}$. This motivates the following definition

Definition 12. *Let V be a meromorphic homogeneous potential on $\mathcal{C}$ of degree -1 such that $c = (1,0)$ is a Darboux point with multiplier -1. We note* $\mathrm{Sp}(\nabla^2 V(c)) = \{2, 1/2(i-1)(i+2)\}$. *Let $k \in \mathbb{N}^*$ be fixed and (VE_k) the k-th order variational equation near the homothetic orbit associated to c. We assume (VE_{k-1}) integrable, so the identity component of the Galois group of (VE_{k-1}) is abelian. We say that* the integrability constraint of (VE_k) is non degenerate *if*

$$\frac{\partial}{\partial \alpha} \underset{t=\infty}{Res} \ (t^2-1)^k (Q_i + \epsilon_i \alpha P_i)^{k+1} \neq 0$$

3.4.3 A rigidity result

We will now prove that non-degeneracy implies a rigidity property. If we take two integrable potentials “close” enough (meaning that enough derivatives on some Darboux point are equals), then they should be equal as proved in Lemma 12 below.

Lemma 12. *Let V_1, V_2 be two integrable meromorphic homogeneous potentials on $\mathcal{C}$ of degree -1 such that $c = (1,0)$ is a Darboux point with multiplier -1. Assume there exists $k_0 \geq 2$ such that integrability constraint of (VE_k) is non degenerate $\forall k \geq k_0$. If*

$$\frac{\partial^{i+j}}{\partial q_1^i \partial q_2^j} V_1(c) = \frac{\partial^{i+j}}{\partial q_1^i \partial q_2^j} V_2(c) \quad \forall (i,j) \text{ such that } i+j \leq k_0$$

then $V = \tilde{V}$.

Proof. We prove this by induction. Assume that all derivatives of V_1 and V_2 are equal up to order $k \geq k_0$. Let us prove that the derivatives of order $k+1$ coincide. Using remark 4, we already know that they coincide except maybe the derivatives

$$d^{(1)}_{k,k+1} = \frac{\partial^{k+1}}{\partial q_2^{k+1}} V_1(c) \qquad d^{(2)}_{k,k+1} = \frac{\partial^{k+1}}{\partial q_2^{k+1}} V_2(c)$$

At first order, the variational equation near the homothetic orbit associated to c has a Galois group whose identity component is abelian (for V_1 and V_2). The eigenvalue $\lambda = \frac{1}{2}(g-1)(g+2)$ is the same for V_1, V_2 as they coincide at least up to order 2. We can then write a solution X of the variational equation (3.9) at order k under the form as in section 3.4.2 page 57

$$\begin{aligned} \sigma_\alpha(y^{(1)}) &= \sigma_\alpha(y_{hom}) + \sigma_\alpha(y_{part_1}) + d^{(1)}_{k,k+1}\sigma_\alpha(y_{part_2}) \\ \sigma_\alpha(y^{(2)}) &= \sigma_\alpha(y_{hom}) + \sigma_\alpha(y_{part_1}) + d^{(2)}_{k,k+1}\sigma_\alpha(y_{part_2}) \end{aligned} \tag{3.10}$$

for respectively V_1, V_2 with

$$y_{part_2}(t) = \int (t^2-1)^k Q_g^{k+1} dt$$

The parts y_{hom} and y_{part_1} can be chosen to be equal as all derivatives of V_1 and V_2 are equal up to order k, and $y_{part_2}(t)$ can be chosen the same as the two potentials V_1, V_2 have the same eigenvalue λ. As V_1, V_2 are both meromorphically integrable, applying the monodromy commutator as before we should obtain

$$\sigma_\alpha(y^{(i)}) - y^{(i)} = 0, \;\; \alpha \in 2i\pi\mathbb{Z}, \;\; i = 1, 2$$

Subtracting these two relations, we get

$$(d^{(1)}_{k,k+1} - d^{(2)}_{k,k+1}) G(\alpha) = 0$$

Now as the integrability constraint of (VE_k) is non degenerate, we have $G(\alpha) \neq 0$ and thus $d^{(1)}_{k,k+1} = d^{(2)}_{k,k+1}$. Thus all derivatives of V_1, V_2 at c up to order $k+1$ coincide. ☐

Lemma 12 allows to prove uniqueness theorems, as if the non degeneracy condition of Definition 12 is satisfied, then a meromorphically integrable potential is completely determined using its derivatives c up to order k. So we now need to look in the literature for meromorphically integrable homogeneous potentials of degree -1 with $c = (1,0)$ as a Darboux point with multiplier -1. The space of series expansion of order k of homogeneous potentials of degree -1 with $c = (1,0)$ and a fixed eigenvalue λ is an affine space $\mathcal{E}$. If all series expansions in $\mathcal{E}$ are series expansions of meromorphically integrable potentials, then this proves that no other exist (as if two meromorphically homogeneous potentials coincide up to order k, they are equal).

For this problem, direct search (e.g. Hietarinta's work in [31]) helps a lot. Still if not enough integrable potentials are found, we only proved that the set of meromorphically integrable potentials is included inside an affine space whose dimension is bounded by $dim(\mathcal{E})$. Remark that this procedure is non constructive as it never allows to find new integrable potentials, but only proves at best that all of them are already found (we could still guess them through their series expansion, but due to computer limitations, we often obtain less than 10 terms).

We now study the cases $\lambda = -1, 0, 2$ because these are the only ones for which we know integrable potentials.

3.4.4 Application to the eigenvalue 0

Lemma 13. *Let V be meromorphic homogeneous potential on $\mathcal{C}$ of degree -1. Assume that $c = (1,0)$ is a Darboux point with multiplier -1, $\mathrm{Sp}(\nabla^2 V(c)) = \{2, 0\}$ and that V is meromorphically integrable. Then $V = 1/q_1$.*

Proof. We just have to use Lemma 12. Let us first check the non degenerescence property. The functions P_1, Q_1 for the eigenvalue 0 are the following

$$P_1 = 1 \qquad Q_1 = \operatorname{arctanh}\left(\frac{1}{t}\right) - \frac{t}{t^2-1}$$

We need to look at the following residue

$$\operatorname*{Res}_{t=\infty} \; (t^2-1)^{k+1}\left(\operatorname{arctanh}\left(\frac{1}{t}\right) + \alpha - \frac{t}{t^2-1}\right)^{k+2}$$

and it should be independent of α. The easiest coefficient to study (and non trivial) seems to be the coefficient in α^{k+1}. Noting it S_k, we find after simplification

$$S_k = \operatorname*{Res}_{t=\infty} \; \left(t^2-1\right)^{k+1}\left((k+2)\operatorname{arctanh}\left(\frac{1}{t}\right) - \frac{kt+2t}{t^2-1}\right)$$

By expanding, we remark that the second term always give a zero residue. Indeed, in the expansion, the fraction simplifies and we get a polynomial. Then, we will compute

$$S_k = \operatorname*{Res}_{t=\infty} \; (k+2)\left(t^2-1\right)^{k+1}\operatorname{arctanh}\left(\frac{1}{t}\right) = \frac{k+2}{2}\int_{-1}^{1}\left(t^2-1\right)^{k+1} dt > 0$$

The last equality is produced with the expansion of $\operatorname{arctanh}\left(\frac{1}{t}\right)$ at infinity. We deduce that

$$S_k \neq 0 \quad \forall k \geq 1$$

Using Lemma 12, we now know that there is a unique potential with $(1,0)$ as Darboux point with multiplier -1 and eigenvalue 0. The potential $1/q_1$ satisfy these conditions, and is integrable because it is invariant by translation. □

Conclusion. So, after rotation, an integrable potential V with a zero eigenvalue near a non degenerate Darboux point corresponds to the potential

$$V = \frac{a}{q_1}, \qquad a \in \mathbb{C}^*$$

3.4.5 Application to the eigenvalue 2

In the case of the eigenvalue 2, we use again the same method. First we will prove that the integrability constraint of (VE_k) is non degenerate at order $k \geq 3$. Thus an integrable potential is uniquely defined by its first three derivatives. In Lemma 15, we find integrable potentials for all possible series expansions, including an exceptional case that appears to coincide after rotation with the "Hietarinta" potential

$$\frac{a(q_1^2+q_2^2)}{(q_1+\epsilon i q_2)^3} + \frac{a}{q_1+\epsilon i q_2}. \tag{3.11}$$

Remark now that in the case of the eigenvalue 2, the Hessian matrix should be diagonalizable for meromorphic integrability. This is a constraint for integrability of first order variational equation, and a complete analysis of the non diagonalizable case at order 1 is given by Duval and Maciejewski in [27]. Let us first prove the non degeneracy hypothesis of Lemma 12.

Lemma 14. *Let V be a meromorphic homogeneous potential on $\mathcal{C}$ of degree -1 such that $c = (1,0)$ is a Darboux point with multiplier -1. Assume that $\operatorname{Sp}(\nabla^2 V(c)) = \{2,2\}$. Then the integrability constraint of the k-th order variational equation (VE_k) is non degenerate for $k \geq 3$, and degenerate at order $k = 2$.*

Proof. We need to look at the following residue

$$\operatorname*{Res}_{t=\infty}\ (t^2-1)^{k+1}\left(-\frac{6t^2-4}{t^2-1}+6t\alpha+6t\operatorname{arctanh}\left(\frac{1}{t}\right)\right)^{k+2}$$

and this residue should be independent of α to prove non degenerescence. We will look at

$$S_k^{(1)}:=\operatorname*{Res}_{t=\infty}\ (t^2-1)^{k+1}t^{k+1}\left(-\frac{6t^2-4}{t^2-1}+6t\operatorname{arctanh}\left(\frac{1}{t}\right)\right)$$

which corresponds to the coefficient in α^{k+1} (after simplifying a non zero factor). But this sequence is not always non zero. We will also need to look at another one

$$S_k^{(2)}:=\operatorname*{Res}_{t=\infty}\ (t^2-1)^{k+1}t^{k}\left(-\frac{6t^2-4}{t^2-1}+6t\operatorname{arctanh}\left(\frac{1}{t}\right)\right)^2$$

Then, we want to prove

$$S_k^{(1)}\neq 0 \text{ or } S_k^{(2)}\neq 0 \qquad \forall k\geq 2$$

More precisely, we will prove that

$$S_{2k}^{(1)}\neq 0 \text{ and } S_{2k+1}^{(2)}\neq 0 \qquad \forall k\geq 1$$

These sequences are D-finite, and as such recurrence formulas can be automatically found and proved for these sequences [38, 40, 39, 75]. Following the creative telescoping approach we found using these algorithms (either Mgfun for Maple, or holonomics for Mathematica) the following recurrences for $S_{2n}^{(1)}$

$$\begin{aligned}64(2n+3)(2n+1)(6n+11)(n+1)^2f(n)-\\ (20736n^5+152064n^4+439200n^3+622752n^2-431784n-116328)f(n+1)+\\ 36(6n+5)(3n+5)(3n+4)(6n+13)(6n+17)f(n+2)\end{aligned}$$

This recurrence can be solved explicitly and gives the formula

$$S_{2n}^{(1)}=-\frac{\pi\Gamma\left(2n+2\right)27^{-n}}{72\Gamma\left(n+\frac{7}{6}\right)\Gamma\left(n+\frac{11}{6}\right)}$$

This expression never vanishes. We do the same for $S_{2n+1}^{(2)}$. We find a third order recurrence and solve it

$$S_{2n+1}^{(2)}=-\frac{\pi 27^{-n}\Gamma\left(2n+3\right)}{3456\Gamma\left(n+\frac{7}{3}\right)\Gamma\left(n+\frac{5}{3}\right)}\sum_{k=0}^{n-1}\left(\frac{(3k+4)\,\Gamma\left(k+5/3\right)\Gamma\left(k+7/3\right)}{(k+1)\,(k+2)\,(2k+3)\,\Gamma\left(k+\frac{13}{6}\right)\Gamma\left(k+\frac{11}{6}\right)}\right)$$

Using this expression, we find that $S_{2n+1}^{(2)}$ never vanish for $n\geq 1$. This prove the non degeneracy condition for order ≥ 3. At order 2, the two formulas vanish. Since in this case the residue is a polynomial in α of degree at most 2, this implies that the residue is constant. So the α derivative is zero and integrability constraint is degenerate. □

We now need to find integrable homogeneous potentials of degree -1 which admit a Darboux point c with spectrum $\{2,2\}$. We already know the potential

$$\frac{a}{q_1}+\frac{b}{q_2},\qquad a,b\in\mathbb{C}^*$$

which is integrable. Computation gives that Darboux points have the eigenvalue 2. So we need to prove that after rotation, all possible 3-rd order derivatives can be produced. As shown below, an exceptional case will be found and will correspond to the Hietarinta potential (3.11).

Lemma 15. *Let V be meromorphic homogeneous potential on $\mathcal{C}$ of degree -1 with $c = (1,0)$ a Darboux point of V with multiplier -1 and $\nabla^2 V(c) = 2I_2$. Then it corresponds after rotation to a potential of the form*

$$\frac{a}{q_1} + \frac{b}{q_2}, \qquad a, b \in \mathbb{C}^* \tag{3.12}$$

except if V admits a series expansion at c of the following form

$$V(c+q) = 1 - q_1 + (q_1^2 + q_2^2) + dq_1^3 + 3dq_1q_2^2 \pm 2idq_2^3 + o(q^3)$$

for which V corresponds after rotation to the Hietarinta potential (3.11)*.*

Proof. We expand V on $c = (1,0)$ which gives

$$V(c+q) = V(c) - q_1 + (q_1^2 + q_2^2) + aq_1^3 + bq_1^2q_2 + cq_1q_2^2 + dq_2^3 + o(q^3)$$

Using remark 4 page 58, we obtain the following values

$$\partial_{1,1,1}V(c) = -6 \quad \partial_{1,1,2}V(c) = 0 \quad \partial_{1,2,2}V(c) = -6$$

Then the series expansion of V on c is always of the form

$$V(c+q) = 1 - q_1 + (q_1^2 + q_2^2) - (q_1^3 + 3q_1q_2^2 + dq_2^3) + o(q^3)$$

where $d \in \mathbb{C}$. We want now prove that such an expansion can correspond to the expansion of the potential (3.12) after rotation. So we will make a rotation of the coordinates q_1, q_2. After rotation, the potentials (3.12) can be written

$$\frac{a}{\mathfrak{c}q_1 - \mathfrak{s}q_2} + \frac{b}{\mathfrak{s}q_1 + \mathfrak{c}q_2}, \qquad \mathfrak{c}^2 + \mathfrak{s}^2 = 1, a, b \in \mathbb{C}^*$$

The condition of admitting a Darboux point at $c = (1,0)$ with multiplier -1 implies that this family of potentials can be written

$$V = \frac{\mathfrak{c}^3}{\mathfrak{c}q_1 - \mathfrak{s}q_2} + \frac{\mathfrak{s}^3}{\mathfrak{c}q_1 + \mathfrak{s}q_2}, \text{ with } \mathfrak{c}^2 + \mathfrak{s}^2 = 1$$

We make series expansion of this expression near $c = (1,0)$ and by identification, we get

$$\frac{-\mathfrak{c}^2 + \mathfrak{s}^2}{\mathfrak{c}\mathfrak{s}} = d, \text{ with } \mathfrak{c}^2 + \mathfrak{s}^2 = 1.$$

This produces the solution

$$\mathfrak{s} = \frac{1}{\sqrt{2}}\sqrt{\frac{4 + d^2 + \sqrt{4d^2 + d^4}}{4 + d^2}}$$

which is valid for $d \neq 2i\epsilon$ with $\epsilon = \pm 1$.

For $d = 2i\epsilon$, there are no solutions, and this is the exceptional case. Let us check that the Hietarinta potential (3.11) corresponds to this case. We will only study the case $\epsilon = +1$ (the case $\epsilon = -1$ is exactly similar). After rotation, we get

$$V = a\frac{(q_1^2 + q_2^2)}{(q_1 + iq_2)^3} + \frac{ab}{q_1 + iq_2}, \qquad a, b \in \mathbb{C}^*$$

The condition of having a Darboux point at $c = (1,0)$ with multiplier -1 gives

$$V = -\frac{1}{2}\frac{q_1^2 + q_2^2}{(q_1 + iq_2)^3} + \frac{3}{2(q_1 + iq_2)}$$

We compute the series expansion at $c = (1,0)$ and this gives exactly the good expansion. Using Lemma 12,14, we know that for each series expansion at order 3, there exists at most one meromorphically integrable potential. We found a meromorphically integrable potential for any possible series expansion at order 3, and so we found all meromorphically integrable potentials with the eigenvalue 2. ∎

3.5 Case of the eigenvalue -1

The case of the eigenvalue -1 is much more difficult because the non degenerescence hypothesis of Lemma 12 does not hold. We need to use a completely different method. We already guess that this case will correspond to the potential $V = r^{-1}$ invariant by rotation. This potential integrates in polar coordinates, which are the action-angles coordinates for this potential. Then, to see some pattern in higher variational equations, it seems to be a good idea to compute all these higher variational equations in polar coordinates. The integrable case $V = r^{-1}$ is quite simple to describe in polar coordinates, as it is the only potential that does not depend on the angle coordinate (the coordinate θ).

So we will first compute higher variational equations up to order 2. Then we will recognize that a strong integrability constraint comes from a particular a simple perturbation, which will allows us to study only a subsystem of these higher variational equations system. We prove in particular that the k-th variational equation possesses invariant vector spaces, and we will find one which is small enough such that the reduced system on this subspace can be more easily analyzed, and not too simple such that the solutions have non-commutative monodromy which put constraints on the derivatives of the potential. The first non trivial integrability condition appears at order 3 with a dilogarithmic term. At higher order we will prove that a non zero $(k+1)$-th derivative $U^{(k+1)}(0) \neq 0$ (the potential being $V = r^{-1}U(\theta)$ in polar coordinates) implies that the Picard Vessiot field of the $2k-1$-th variational equation contains a dilogarithmic term.

Proposition 6. *(proved page 69) Let V be meromorphic homogeneous potential on $\mathcal{C}$ of degree -1 with $c = (1,0)$ a Darboux point of V with multiplier -1. Assume that $\mathrm{Sp}(\nabla^2 V(c)) = \{2, -1\}$. If V is integrable, then $V = r^{-1}$.*

3.5.1 Variational equation of order 2

Before proving Proposition 6, let us first look only at order 2.

Lemma 16. *Let $V = r^{-1}U(\theta)$ be a meromorphic homogeneous potential on $\mathcal{C}$ of degree -1 with $c = (1,0)$ a Darboux point of V with multiplier -1. Assume that $\mathrm{Sp}(\nabla^2 V(c)) = \{2, -1\}$. The Galois group of the second order variational equation near the homothetic orbit associated to c of the Hamiltonian field in polar coordinates is $\mathbb{C}^2$ if $U^{(3)}(0) \neq 0$ and $\mathbb{C}$ if $U^{(3)}(0) = 0$.*

Proof. The potential $V = r^{-1}U(\theta)$ gives the following differential equations in polar coordinates

$$\ddot{r} - r\dot{\theta}^2 = -\frac{1}{r^2}U(\theta), \qquad \ddot{\theta} + 2\frac{\dot{r}}{r}\dot{\theta} = \frac{1}{r^3}U'(\theta). \tag{3.13}$$

Let us linearize this equation near a homothetic orbit corresponding to the critical point 0 of U. We assume moreover that $U''(0) = 0$, which corresponds to $\mathrm{Sp}(\nabla^2 V(c)) = \{2, -1\}$. We get at first order

$$\ddot{r} = \frac{2U(0)}{\phi^3}r, \qquad \ddot{\theta} + 2\frac{\dot{\phi}}{\phi}\dot{\theta} = \frac{U''(0)}{\phi^3}\theta = 0.$$

We choose the normalization of c such that the multiplier is -1 and we make the variable change $\dot{\phi}/\sqrt{2} \longrightarrow t$ which gives

$$\frac{1}{2}\left(t^2 - 1\right)\ddot{r} + 2\dot{r}t - 2\,r = 0, \qquad \frac{1}{2}\left(t^2 - 1\right)\ddot{\theta} = 0.$$

Of course these equations are integrable (because they correspond to an integrable case of Theorem 13 and the solutions are

$$r(t) = C_1 P_2 + C_2 Q_2, \qquad \theta(t) = C_3 t + C_4$$

Now we take a look at second-order variational equations. As in equation (3.8), we first compute the series expansion of order 2 of equation (3.13) at $\dot{r} = \dot{\phi}, \dot{\theta} = 0, r = \phi, \theta = 0$

$$\begin{aligned} \ddot{r} - \phi\dot{\theta}^2 &= \frac{2}{\phi^3}r - \frac{3}{\phi^4}r^2 \\ \ddot{\theta} + 2\frac{\dot{\phi}}{\phi}\dot{\theta} + \frac{2}{\phi}\dot{r}\dot{\theta} - \frac{2\dot{\phi}}{\phi^2}r\dot{\theta} &= \frac{1}{\phi^3}U^{(3)}(0)\theta^2 \end{aligned} \tag{3.14}$$

Using again the same procedure as in page 58, the second order variational equation may be written (after variable change $\dot{\phi}/\sqrt{2} \longrightarrow t$)

$$\begin{aligned} \frac{1}{2}(t^2-1)\ddot{r}_2 + 2t\dot{r}_2 - \frac{1}{2}\dot{\theta}_1^2 &= 2r_2 - 3(t^2-1)r_1^2 \\ \frac{1}{2}\ddot{\theta}_2 + 2tr_1\dot{\theta}_1 + (t^2-1)\dot{r}_1\dot{\theta}_1 &= \frac{1}{2}\frac{1}{t^2-1}U^{(3)}(0)\theta_1^2 \end{aligned} \tag{3.15}$$

where r_1, θ_1 are solutions of the first order variational equation. The first equation of (3.15) integrates because the non homogeneous term

$$-\frac{1}{2}\dot{\theta}_1^2 = -\frac{1}{2}C_3^2$$

corresponds to a particular solution to r_2 of the form

$$\int (t^2-1)Q_2^3 dt$$

(where Q_2 is defined page 56) and whose monodromy is commutative. For the second equation of (3.15), we find

$$\frac{1}{2}\ddot{\theta}_2 + 2t(C_1P_2 + C_2Q_2)C_3 + (t^2-1)(C_1\dot{P}_2 + C_2\dot{Q}_2)C_3 = \frac{1}{2}\frac{1}{t^2-1}U^{(3)}(0)(C_3t + C_4)^2$$

The solution can be written as

$$2\iint -2t(C_1P_2 + C_2Q_2)C_3 - (t^2-1)(C_1\dot{P}_2 + C_2\dot{Q}_2)C_3 + \frac{1}{2}U^{(3)}(0)\frac{(C_3t+C_4)^2}{t^2-1}dt^2$$

We have the following formulas

$$P_2 = 4t \quad Q_2 = \frac{3}{8}t \operatorname{arctanh}\left(\frac{1}{t}\right) + \frac{\frac{1}{4} - \frac{3}{8}t^2}{t^2-1}$$

The terms in P_2 are polynomials and integrate well. For the terms in Q_2, we find the following expression (up to c integration constants)

$$\iint -2tQ_2 - (t^2-1)\dot{Q}_2 dtdt = \iint \frac{3}{8}(3t^2-1)\operatorname{arctanh}\left(\frac{1}{t}\right) - \frac{9}{8}tdtdt = \frac{3}{32}(t^2-1)^2 \operatorname{arctanh}\left(\frac{1}{t}\right) - \frac{3}{32}t^3$$

Then the only term left is

$$\iint \frac{1}{2}U^{(3)}(0)\frac{(C_3t+C_4)^2}{t^2-1}dtdt \in \mathbb{C}\left[t, \operatorname{arctanh}\left(\frac{1}{t}\right), \ln\left(t^2-1\right)\right] \tag{3.16}$$

Then the second-order variational equation is always integrable, and moreover we have computed its Galois group

- If $U^{(3)}(0) \neq 0$ then the Galois group of (3.15) is $\mathbb{C}^2$
- If $U^{(3)}(0) = 0$ then the Galois group of (3.15) is $\mathbb{C}$

□

3.5.2 Degeneracy of higher variational equations

Let us now look at the non-degeneracy property of variational equation. In the case of eigenvalue -1, we have $\epsilon_0 = 0$ (see page 56) because the first order variational equation has two independent rational solutions

$$P_0 = t(t^2-1)^{-1} \qquad Q_0 = (t^2-1)^{-1}$$

Still we could think that a similar condition to 12 of the non-degeneracy could still apply. The term corresponding to the highest order derivative of the potential is given by

$$\int (t^2-1)^k (aQ_0+bP_0)^{k+1} dtdt \in \mathbb{C}\left[t, \operatorname{arctanh}\left(\frac{1}{t}\right), \ln\left(t^2-1\right)\right]$$

and thus this term has always a commutative monodromy. Computing variational equations of the Hamiltonian field in polar coordinate does not help either.

Proposition 7. *Let V be meromorphic homogeneous potential on $\mathcal{C}$ of degree -1 with $c=(1,0)$ a Darboux point of V with multiplier -1. Assume that $\mathrm{Sp}(\nabla^2 V(c)) = \{2,-1\}$. Assume that $U^{(i)}(0) = 0\ \forall i = 1\ldots k$ then the fact that the identity component of the Galois group of the k-th order variational equation is abelian or not does not depend on the value of $U^{(k+1)}(0)$.*

Proof. The case of order 2 corresponds to the previous proof. Let us look now at order k. We pick in the equations the non homogeneous terms where $U^{(k+1)}(0)$ appear. The only equation where such term appear is the following

$$\frac{1}{2}\ddot{\theta}_k = \frac{1}{k!}\frac{U^{(k+1)}(0)}{t^2-1}\theta_1^k \tag{3.17}$$

where θ_1 is solution of the first order variational equation (we have removed all non homogeneous terms in which $U^{(k+1)}(0)$ does not appear). The solution for θ_1 is $\theta_1(t) = at+b$, and then substituting this expression, we obtain that the solution of equation (3.17) is of the form

$$\theta_k(t) = \frac{2U^{(k+1)}(0)}{k!}\iint \frac{(at+b)^k}{t^2-1} dtdt \in \mathbb{C}\left[t, \operatorname{arctanh}\left(\frac{1}{t}\right), \ln\left(t^2-1\right)\right]$$

which can be checked using recursive integration by parts. This term has then a commutative monodromy, and then the fact that the identity component of the Galois group is abelian or not does not depend on the value of $U^{(k+1)}(0)$. ☐

Remark 5. *We remark that the integral*

$$\iint \frac{(at+b)^k}{t^2-1} dtdt$$

does not belong to the Picard-Vessiot field of the first order variational equation (which is $\mathbb{C}\left[t, \operatorname{arctanh}\left(\frac{1}{t}\right)\right]$). So the Picard-Vessiot field of the k-order variational equation is generically larger (when $U^{(k+1)}(0) \neq 0$), and the Galois group becomes at least $\mathbb{C}^2$. But this does not give us any integrability condition, as the Galois group could still be abelian (with a higher dimension). This is precisely why this case is particularly difficult. We cannot use non degenerescence properties, and so we need to keep these unknown derivatives $U^{(i)}(0)$ as parameters and go higher in the order of variational equations.

3.5.3 An invariant subspace of the $(2k-1)$-th order variational equation

Let us first look at variational equation of order $2k-1$. We compute the series expansion of equation (3.13) at $\dot{r} = \dot{\phi}, \dot{\theta} = 0, r = \phi, \theta = 0$ of order $k \geq 3$

$$\begin{gathered} \ddot{r} - \phi\dot{\theta}^2 - r\dot{\theta}^2 = \sum_{i=1}^{2k-1} \frac{(-1)^{i+1}(i+1)}{\phi^{i+2}} r^i + \sum_{i=k+1}^{2k-1} \sum_{j=0}^{i-k-1} \frac{(-1)^{j+1}(j+1)}{\phi^{j+2}(i-j)!} U^{(i-j)}(0) r^j \theta^{i-j} \\ \ddot{\theta} + \sum_{i=0}^{2k-2} \frac{2(-1)^i \dot{\phi}}{\phi^{i+1}} r^i \dot{\theta} + \sum_{i=0}^{2k-3} \frac{2(-1)^i}{\phi^{i+1}} \dot{r} r^i \dot{\theta} = \sum_{i=k}^{2k-1} \sum_{j=0}^{i-k} \frac{(-1)^j (j+1)(j+2)}{2\phi^{j+3}(i-j)!} U^{(i-j+1)}(0) r^j \theta^{i-j} \end{gathered} \tag{3.18}$$

The $U^{(i)}(0)$ $i = k+1 \dots 2k$ are parameters. Using these series expansions, let us build now the $2k-1$-th variational equation. As explained in section 4.2., we use the substitution

$$y_{i,j,l,m} = \dot{r}^i \dot{\theta}^j r^l \theta^m$$

For $i+j+l+m = 2k-1$, the differential equation system for the $y_{i,j,l,m}$ is the $2k-1$-symmetric product of the first order variational equation. As we do not want to compute the complete solution of the $2k-1$-th variational equation, we will only compute one well chosen solution. To simplify the equation, we will first build an invariant vector space, and then solve the variational on this subspace (and in fact only for some of the variables).

Lemma 17. *Assume that $U(0) = 1, U^{(i)}(0) = 0 \;\; \forall i = 1 \dots k$. Then the vector space $\mathcal{W}$ given by the conditions*

$$y_{i,j,l,m} = 0 \quad \forall i,j,l,m \text{ such that } j+m \geq k,\; i+l \geq 1 \quad (C_1)$$

$$y_{i,j,l,m} = 0 \quad \forall i,j,l,m \text{ such that } j \geq 1,\; i+l \geq 1 \quad (C_2)$$

$$y_{i,j,l,m} = 0 \quad \forall i,j,l,m \text{ such that } j \geq 2 \quad (C_3)$$

$$y_{i,j,l,m} = 0 \quad \forall i,j,l,m \text{ such that } j \geq 1,\; j+m \geq k+1 \quad (C_4)$$

is an invariant vector space of the $2k-1$-th variational equation.

We can ask why to consider such vector space. Here we want to build an analog of normal variational equation (which is properly defined only at order 1). In particular, we want to suppress all terms in $\dot{r}, r$ in the second equation of (3.18). This is the reason of the conditions $(C_1), (C_2)$. The conditions $(C_3), (C_4)$ are necessary to $\mathcal{W}$ to be invariant.

The reason of conditions $(C_3), (C_4)$ is that we need to suppress the term in $\dot{\theta}^2$ corresponding to centrifugal force. Physically, this means that the perturbation we are interested in will correspond to very small values of $\dot{theta}$, and if $\dot{\theta}$ is non-zero, θ should be small also (condition (C_4)). The condition (C_2) implies that the Coriolis force coming from perturbations of order ≥ 2 is negligible.

Proof. We only need to prove that the derivative in time of these $y_{i,j,l,m} = 0$ only involve these $y_{i,j,l,m}$ (because then the differential equation being linear, 0 will be solution of this subsystem). Let us derive $\dot{r}^i \dot{\theta}^j r^l \theta^m$. Using Leibniz derivation rule, this produces 4 terms

$$m\dot{r}^i \dot{\theta}^{j+1} r^l \theta^{m-1} + l\dot{r}^{i+1} \dot{\theta}^j r^{l-1} \theta^m + i\ddot{r}\dot{r}^{i-1} \dot{\theta}^j r^l \theta^m + j\ddot{\theta}\dot{r}^i \dot{\theta}^{j-1} r^l \theta^m \tag{3.19}$$

The two first terms still satisfy the condition, and for the two last ones we have to replace $\ddot{r}, \ddot{\theta}$ using relation (3.18). For $\ddot{r}$

- If the condition (C_1) is satisfied, the two first terms of the first equation (3.18) will produce terms with degree in $\dot{\theta}$ at least 2, and thus satisfying condition (C_3). In the right part, the only potentially problematic terms in the sums (of the first equation (3.18)) are in the second one for $j=0$

$$\sum_{i=k+1}^{2k-1} -\frac{1}{\phi^2 i!} U^{(i)}(0)\theta^i$$

As $j+m \geq k$, after multiplication we get terms of degree in $(\theta,\dot{\theta}) \geq 2k+1$, and so are discarded because we study variational equation of order $2k-1$.

- If the condition (C_2) is satisfied, the two first terms of the first equation (3.18) will produce terms with degree in $\dot{\theta}$ at least 3, and thus satisfying condition (C_3). In the right part, the only potentially problematic terms are in the second sum (of the first equation (3.18)) for $j=0$. These will produce terms with degree in θ at least $k+1$ and degree in $\dot{\theta}$ at least 1, so satisfying condition (C_4).

- If the condition (C_3) is satisfied, the degree in $\dot{\theta}$ cannot decrease, and so the three terms satisfy condition (C_3).

- If the condition (C_4) is satisfied, the degree in $\dot{\theta}$ and in θ cannot decrease, and so the three terms satisfy condition (C_4).

For $\ddot{\theta}$

- If the condition (C_1) is satisfied, the two first terms of the second equation (3.18) contains $\dot{\theta}$, and thus condition (C_1) is still satisfied. In the right sum, problematic terms would be those containing no $\dot{\theta},\theta$, corresponding to $i=j$, but these are out of the interval of summation.

- If the condition (C_2) is satisfied, the two first terms of the second equation (3.18) will produce terms with degree in $\dot{\theta}$ at least 1 and degree in $r,\dot{r}$ at least 1, and thus satisfying condition (C_2). The right sum will produce terms with degree in θ at least k and degree in $r,\dot{r}$ at least 1, so satisfying condition (C_1).

- If the condition (C_3) is satisfied, the two first terms of the second equation (3.18) will produce terms with degree in $\dot{\theta}$ at least 2, thus satisfying condition (C_3). The right sum produces terms of degree in $\dot{\theta}$ at least 1 and degree in θ at least k, so satisfying condition (C_4).

- If the condition (C_4) is satisfied, the two first terms of the second equation (3.18) will produce terms with degree in $\dot{\theta}$ at least 1 and degree in $\theta,\dot{\theta}$ at least $k+1$, thus satisfying condition (C_4). The right sum produces terms of degree in $\theta,\dot{\theta}$ at least $2k$, and thus which are suppressed because we study only the $(2k-1)$-th variational equation

So this subspace is invariant. □

We can now formally remove the corresponding terms in equation (3.18)

$$\ddot{r} = \sum_{i=1}^{2k-1} \frac{(-1)^{i+1}(i+1)}{\phi^{i+2}} r^i + \sum_{i=k+1}^{2k-1} -\frac{U^{(i)}(0)}{\phi^2 i!}\theta^i \qquad \ddot{\theta} + 2\frac{\dot{\phi}}{\phi}\dot{\theta} = \sum_{i=k}^{2k-1} \frac{U^{(i+1)}(0)}{\phi^3 i!}\theta^i \tag{3.20}$$

As we see, in the second equation, terms containing $(\dot{r},r)$ no longer appear.

Remark 6. *One of the interest of the space $\mathcal{W}$ is that is dimension is much lower. Moreover, in the following, we will only be interested by the second equation of* (3.20)*, and thus reasoning in the dimension of $\mathcal{W}' = \mathcal{W} \cap \{y_{i,j,l,m} = 0,\ \forall i + l \geq 1\}$. By guessing, we find*

$$\dim(VE_{2k-1}) = \frac{1}{6}(2k+3)(k+4)(2k^2+11k+17) \qquad \dim \mathcal{W}' = 3k-1$$

$$\dim \mathcal{W} = 4 \prod_{s=0}^{k} \left(\frac{7s^3+51s^2+134s+114}{7s^3+30s^2+53s+24} \right) \sim \frac{7}{6}k^3$$

Clearly, studying the second equation of (3.20) *on $\mathcal{W}$ and computing solutions for variables in $\mathcal{W}'$ will be much easier than the complete system.*

3.5.4 Proof of Proposition 6

Proof. To prove Proposition 6, it is only necessary to prove that $U^{(i)}(0) = 0 \quad \forall i \in \mathbb{N}^*$. It is already proven for $i = 1, 2$ by hypothesis. Let us prove this by recurrence. Assume that $U^{(i)}(0) = 0\ \forall i = 1 \dots k$. We want to prove that $U^{(k+1)}(0) = 0$.

Let us now study the $(2k-1)$-th order variational equation, and in particular on the invariant subspace $\mathcal{W}$ given by Lemma 17. We may now try to find a solution of the variational equation on this invariant subspace. We will only compute closed form expression for some of the unknowns (those who appear in the second equation of (3.20)). We have

$$\dot{y}_{0,0,0,m} = m y_{0,1,0,m-1} = 0 \qquad \forall m \geq k+1$$

We choose the solution $y_{0,0,0,m} = 0,\ m = k+1 \dots 2k-2$ and $y_{0,0,0,2k-1} = 1$. We also find the differential equation

$$\dot{y}_{0,0,0,k} = k y_{0,1,0,k-1} \qquad \dot{y}_{0,1,0,k-1} = -\frac{2\dot{\phi}}{\phi} y_{0,1,0,k-1} + \frac{U^{(k+1)}(0)}{\phi^3 k!} y_{0,0,0,2k-1}$$

Substituting $y_{0,0,0,2k-1}$ by its expression, we get

$$\ddot{y}_{0,0,0,k} + \frac{2\dot{\phi}}{\phi}\dot{y}_{0,0,0,k} = \frac{U^{(k+1)}(0)}{\phi^3 (k-1)!}$$

The other interesting equation of the variational equation is

$$\ddot{y}_{0,0,0,1} + \frac{2\dot{\phi}}{\phi}\dot{y}_{0,0,0,1} = \frac{U^{(k+1)}(0)}{\phi^3 k!} y_{0,0,0,k} + \frac{U^{(2k)}(0)}{\phi^3 (2k-1)!}$$

We now make the variable change $\dot{\phi}/\sqrt{2} \longrightarrow t$. This produces the equations

$$\begin{aligned} \frac{1}{2}(t^2-1)\ddot{y}_{0,0,0,k} &= \frac{U^{(k+1)}(0)}{(k-1)!} \\ \frac{1}{2}(t^2-1)\ddot{y}_{0,0,0,1} &= \frac{U^{(k+1)}(0)}{k!} y_{0,0,0,k} + \frac{U^{(2k)}(0)}{(2k-1)!} \end{aligned} \tag{3.21}$$

We can now solve them. We find $y_{0,0,0,k} =$

$$\iint \frac{2U^{(k+1)}(0)}{(k-1)!(t^2-1)} dt dt = -\frac{2U^{(k+1)}(0)}{(k-1)!} \left(t \operatorname{arctanh}\left(\frac{1}{t}\right) + \frac{1}{2}\ln\left(t^2-1\right) \right)$$

and then

$$y_{0,0,0,1} = -\frac{4U^{(k+1)}(0)^2}{k!(k-1)!}\iint \frac{1}{t^2-1}\left(t\operatorname{arctanh}\left(\frac{1}{t}\right)+\frac{1}{2}\ln(t^2-1)\right)dtdt-$$
$$\frac{2U^{(2k)}(0)}{(2k-1)!}\left(t\operatorname{arctanh}\left(\frac{1}{t}\right)+\frac{1}{2}\ln\left(t^2-1\right)\right)=$$
$$\frac{2U^{(k+1)}(0)^2}{k!(k-1)!}\left((t+1)(\ln(t-1)+1)\ln(t+1)-((2\ln 2+1)t-1)\ln(t-1)+2t\operatorname{dilog}\left(\frac{t+1}{2}\right)\right)-$$
$$\frac{2U^{(2k)}(0)}{(2k-1)!}\left(t\operatorname{arctanh}\left(\frac{1}{t}\right)+\frac{1}{2}\ln\left(t^2-1\right)\right)$$

All the terms are in $\mathbb{C}[t, \operatorname{arctanh}\left(\frac{1}{t}\right), \ln\left(t^2-1\right)]$ except one, the dilogarithmic term

$$\operatorname{dilog}\left(\frac{t+1}{2}\right)=\int \frac{\ln(t+1)-\ln 2}{1-t}dt$$

The dilogarithm has a non commutative monodromy (see [5]). As expected, the term in $U^{(2k)}(0)$ has a commutative monodromy. So the integrability constraint is that the dilogarithmic term should not appear. Then a necessary integrability constraint is that $U^{(k+1)}(0)=0$, which completes the recurrence. The function U is meromorphic, all derivatives of U are zero

$$U^{(k)}(0)=0\ \forall k\in\mathbb{N}^*,$$

and so this implies that U is constant. Then $V=r^{-1}$. □

To conclude, we have found all meromorphically integrable meromorphic homogeneous potentials on $\mathcal{C}$ which have a Darboux point with eigenvalues -1 (Proposition 6), 0 (Lemma 13) and 2 (Lemma 15). This implies Theorem 11.

3.6 The other eigenvalues: $5, 9, 14, 20, \ldots$

To find all integrable potentials, we would need to study all the other possible eigenvalues. But for these larger eigenvalues, no integrable homogeneous potential of degree -1 is known. Then we can assume that such potentials do not exists, and we can also make a stronger assumption that at some order k, the k-th variational equation has never a Galois group whose identity component is abelian (for any choice of derivatives of V at c of order ≥ 3). First of all, we will give some application theorems for the non degenerescence property. This property cannot be checked for all possibilities for the moment, but with an axi-symmetric assumption, such property can be proved.

Theorem 14. *Let V be meromorphic homogeneous potential on $\mathcal{C}$ of degree -1 with $V=\frac{1}{r}U(\theta)$ in polar coordinates. The set of meromorphically integrable potentials $V=r^{-1}U(\theta)$ with $U(-\theta)=U(\theta)$ and $U(0)=1$ is at most countable.*

Remark 7. *More precisely, we will prove that for each eigenvalue, there is at most one axi-symmetric meromorphically integrable potential. We say nothing about their existence, and we only know them for $\lambda=-1,0,2$ which are respectively*

$$W_0=\frac{1}{r}\qquad W_1=\frac{1}{r}\frac{1}{\cos(\theta)}\qquad W_2=\frac{1}{r}\frac{\cos(\theta)}{\cos(2\theta)}$$

Proof. We just need to prove the non degenerescence property for odd orders. Indeed, for even orders, we use Euler relation which gives us all derivatives except one, the derivative in the normal direction to the straight line $\theta=0$ (see remark 4). But for the variational equation of even order k, this maximal order derivative is then of odd order $k+1$. This derivative is then automatically 0 because we assume the symmetry. The non degenerescence is written

$$\frac{\partial}{\partial\alpha}\operatorname*{Res}_{t=\infty}\ (t^2-1)^k(Q_n+\alpha\epsilon_n P_n)^{k+1}\neq 0$$

We look at coefficient α^k of the above residue, i.e :

$$\epsilon_n^k(k+1)\operatorname*{Res}_{t=\infty}\ (t^2-1)^k Q_n P_n^k=\frac{1}{2}\epsilon_n^k(k+1)\int_{-1}^{1} P_n^{k+1}dt$$

by using the Taylor expansion of arctanh $\left(\frac{1}{t}\right)$ at infinity and recognizing that this sum can be written as this integral. The integer k is odd, the polynomials P_n are never identically 0, then this coefficient never vanishes. This proves non degenerescence, and thus unicity thanks to Lemma 12. □

Conjecture 1. *Let V be meromorphic homogeneous potential on $\mathcal{C}$ of degree -1 with $V=\frac{1}{r}U(\theta)$ in polar coordinates. We assume there exists θ_0 such that $U'(\theta_0)=0$ and*

$$\frac{U''(\theta_0)}{U(\theta_0)}-1\in\left\{\frac{1}{2}(n-1)(n+2),\ k\in\mathbb{N},\ n\geq 3\right\}$$

Then there exists a $k\in\mathbb{N}^*$ such that the variational equation at order k is not integrable.

Conjecture 2. *Let V be meromorphic homogeneous potential on $\mathcal{C}$ of degree -1 with $V=\frac{1}{r}U(\theta)$ in polar coordinates. We assume there exists θ_0 such that $U'(\theta_0)=0$ and $U(\theta_0)\neq 0$. If*

$$\frac{U''(\theta_0)}{U(\theta_0)}-1=\frac{1}{2}(n-1)(n+2)\quad n \textit{ odd } n\geq 3$$

then the 5-th variational equation is not integrable. If

$$\frac{U''(\theta_0)}{U(\theta_0)}-1=\frac{1}{2}(n-1)(n+2)\quad n \textit{ even } n\geq 4$$

then the 7-th variational equation is not integrable.

The second conjecture seems to be extremely difficult. Solving it would classify completely integrable homogeneous potentials of degree -1 in the plane, and would probably allow with some generalization for other degrees to close completely the search of integrable homogeneous potentials (with at least some assumption on Darboux points). A partial proof up to the 5-th variational equation would lead to classification of axi-symmetric integrable potentials and so it would imply for example that in Theorem 14, there are only 3 axi-symmetric meromorphically integrable potentials, and thus that all of them are known. This would also lead to numerous theorems in higher dimension for potentials having discrete symmetry groups. We will prove these conjectures for $n=3,4,5,6$, and give all corresponding expansions of V integrable at order k (with $k\leq 4$ or ≤ 6 depending on parity). This proves by the way the nonexistence of the potentials associated to eigenvalues $\lambda=5,9,14,20$ of Theorem 14. Very concretely, this means that the eigenvalues $\lambda=5,9,14,20$ in Morales-Ramis in [53] for homogeneity degree $k=-1$ are never possible for a planar homogeneous potential, and thus can be removed. Moreover, we present an algorithm able to prove (or disprove) this conjecture for an arbitrary fixed number of eigenvalues.

Proof. (Partial proof of conjectures 1 and 2) We will assume that $c=(1,0)$ is a Darboux point and that V has been normalized such that the multiplier of c is -1. We note $\operatorname{Sp}(\nabla^2 V(c))=$

$\{2,\lambda\}$. The corresponding angle in polar coordinates is $\theta = 0$. We begin by odd index eigenvalues, because we will just have to look at order 5. This correspond to $\lambda = 5, 14$. We know that the 3-th derivative in q_2 should be zero for integrability at order 2. The Euler relation for V gives the following expansion near c

$$V = -q_1 + \frac{1}{2}(2q_1^2 + \lambda q_2^2) - q_1^3 - \frac{3}{2}\lambda q_1 q_2^2 + q_1^4 + 3\lambda q_1^2 q_2^2 + z q_2^4 + o(q^4)$$

where z is an unknown coefficient. This is this coefficient that our program will determine. To make more readable the results, we will write the solution with an expansion in polar coordinates (even if the variational equations are computed in Cartesian coordinates). We have for the moment (let us recall that the k-th order variational equation gives constraints on the $k+1$-th derivatives)

$$V_3 = \frac{1}{r}\left(1 + 3\theta^2 + o(\theta^3)\right) \qquad V_5 = \frac{1}{r}\left(1 + \frac{15}{2}\theta^2 + o(\theta^3)\right)$$

We want now to compute variational equations to very high order to see if we can always be integrable at high order by choosing good values for the derivatives. Let us build a general algorithm for computation of higher variational equation. We already know that we will need to compute the differential systems associated to

$$y_{i,j,k,l} = \dot{X_1}^i \dot{X_2}^j X_1^k X_2^l$$

This produces linear differential equations in $y_{i,j,k,l}$ with $i+j+k+l \leq n$ where n is the order of the expansion. In Maple style, this formula can be written

$$\begin{gathered}\dot{y}_{i,j,k,l} = k y_{i+1,j,k-1,l} + l y_{i,j+1,k,l-1} - \frac{4ity_{i,j,k,l}}{t^2-1} - \frac{4jty_{i,j,k,l}}{t^2-1} + \\ i \operatorname{add}(\operatorname{add}(\operatorname{coeff}(\operatorname{coeff}(W_1, q_1^n), q_2^p) y_{i-1,j,k+n,l+p}, n = 0\ldots\infty), p = 0\ldots\infty) + \\ j \operatorname{add}(\operatorname{add}(\operatorname{coeff}(\operatorname{coeff}(W_2, q_1^n), q_2^p) y_{i,j-1,k+n,l+p}, n = 0\ldots\infty), p = 0\ldots\infty)\end{gathered}$$

where

$$W_1 = \operatorname{subs}\left(q_1 = (t^2-1)q_1, q_2 = (t^2-1)q_2, \frac{2}{(t^2-1)^2}\partial_{q_1} V\right)$$
$$W_2 = \operatorname{subs}\left(q_1 = (t^2-1)q_1, q_2 = (t^2-1)q_2, \frac{2}{(t^2-1)^2}\partial_{q_2} V\right)$$

To simplify, we look at the system with $i+k$ and $j+l$ fixed. The $y_{i,j,k,l}$ in the equations have all $i+k$ and $j+l$ equal or higher. This produce a block triangularization of our system.

Practical computation

We already know the solutions for $i+j+k+l = n$. We can then take

$$y_{i,j,k,l} = \dot{Q_p}^j Q_p^l \quad \text{if } i = k = 0$$

$$y_{i,j,k,l} = 0 \quad \text{otherwise}$$

This choice correspond to a normal perturbation, with as much as possible functions Q (this is because they are more complicated, so we can expect more integrability constraints from them). By linearity, we also can make the even better choice

$$y_{i,j,k,l} = \operatorname{coeff}((\dot{Q}_p + \epsilon_p \alpha \dot{P}_p)^j (Q_p + \epsilon_p \alpha P_p)^l, \alpha^m) \quad \text{if } i = k = 0$$

$$y_{i,j,k,l} = 0 \quad \text{otherwise}$$

which select in advance the coefficient in α in the final residue to compute. This allows to simplify the computation by doing it separately. Another difficulty is that even if the system should always have a solvable Galois group, it is very difficult to solve. We still remark that any integral of a rational fraction with poles at $-1, 1$ can be expressed with $\operatorname{arctanh}(\frac{1}{t}), \ln(t^2-1)$, and thus if the Galois group stays abelian, the Picard-Vessiot extension should at most contains these functions. As only the $\operatorname{arctanh}(\frac{1}{t})$ seems to appear in the computations (with the exception of the eigenvalue $\lambda = -1$), we will then just search solutions of the form

$$y_{i,j,k,l} \in \mathbb{C}(t)\left[\operatorname{arctanh}\left(\frac{1}{t}\right)\right]$$

with the degree in arctanh less than m, except for the equation in $i = k = 0$ and $j + l = 1$ (such a search can be done in practice thanks to Barkatou algorithm [10], and could probably be enhanced by [12] and control of singularities). We also assume that the degree in arctanh does not grow. The question of integrability at order n arrives only when solving the last equation. This last equation is solved with standard method of variation of constants, and the integrability constraint is established using Lemma 11. Let us precise that all these conditions are sufficient to assure the termination of the algorithm, but are in no way necessary. In all cases, if the algorithm gives a solution, the solution is always valid. One important precision is that we need to use **only one solution** (although well chosen).

We find the following expansions

$$V_3 = \frac{1}{r}\left(1 + 3\theta^2 + \frac{125}{12}\theta^4 + o(\theta^5)\right) \qquad V_5 = \frac{1}{r}\left(1 + \frac{15}{2}\theta^2 + \frac{374495}{5352}\theta^4 + o(\theta^5)\right)$$

and that V_3, V_5 are never integrable at order 5. To prove this, the easiest way is to consider the case $m = 1$, and $m = 3$, and this give two incompatible constraints for the 6-th derivative.

In the case of the even index, we have the following expansions at first order

$$V_4 = \frac{1}{r}\left(1 + 5\theta^2 + b\theta^3 + o(\theta^3)\right) \qquad V_6 = \frac{1}{r}\left(1 + \frac{21}{2}\theta^2 + b\theta^3 + o(\theta^3)\right)$$

We have here a free parameter b. This is the main reason of the greater difficulty. When we arrive at order 5, we also get two incompatible conditions for the 6-th derivative, except for some well chosen b. We find then the following expansions (for $r = 1$)

$$\frac{1}{5!}\partial_\theta^5 V_4 = \frac{363467}{4824000}b^3 + \frac{112035}{8576}b$$

$$\frac{1}{6!}\partial_\theta^6 V_4 = \frac{216926052083}{10224685080000}b^4 + \frac{279352141289}{54531653760}b^2 + \frac{4715685295}{24563808}$$

with

$$R_4(b) = \frac{158469311}{97702546320000}b^4 + \frac{372429603}{868467078400}b^2 + \frac{45927}{2729312} = 0$$

$$\frac{1}{5!}\partial_\theta^5 V_6 = \frac{68250852673}{4257725150000}b^3 + \frac{98831601}{3475694}b$$

$$\frac{1}{6!}\partial_\theta^6 V_6 = \frac{10915637473609903}{5190230823727250000}b^4 + \frac{6605928379884787}{1271076936423000}b^2 + \frac{19638863047783}{10039110960}$$

with

$$R_6(b) = \frac{198715111646995383}{27724359406485352625000000}b^4 + \frac{1448561702310687}{7921245544710100750}b^2 + \frac{270431334600}{3128145145507} = 0$$

So for each expansions for eigenvalues $9, 20$ there are exactly 4 possible expansion integrable at order 5. We remark that this set of possible expansions is invariant by the transformation $\theta \longrightarrow -\theta$ which is expected because this transformation is a symmetry and does not change integrability or non integrability of the potential. For higher orders, we need then to assume that $R_i(b) = 0$. The explicit expression of b is too complicated to do efficiently the computations at higher order. So, we first remark that b appear only in the non homogeneous part of the linear differentials systems we have to solve. This means that all our computations can be done in

$$\mathbb{Q}(b)\left[t, \frac{1}{t^2-1}, \operatorname{arctanh}\left(\frac{1}{t}\right)\right] \simeq \mathbb{Q}\left[X, t, \frac{1}{t^2-1}, \operatorname{arctanh}\left(\frac{1}{t}\right)\right] / (R_i(X))$$

if R_i is irreducible (which in our case has always been true but it can of course be tested for each eigenvalue). We then just have to write for each differential system the non homogeneous part as a linear combination of $1, b, b^2, b^3$, then solve it for each term and then sum. Indeed, the solutions of each term will be in (at least we assume it with our assumption on Galois groups)

$$\mathbb{Q}\left[t, \frac{1}{t^2-1}, \operatorname{arctanh}\left(\frac{1}{t}\right)\right]$$

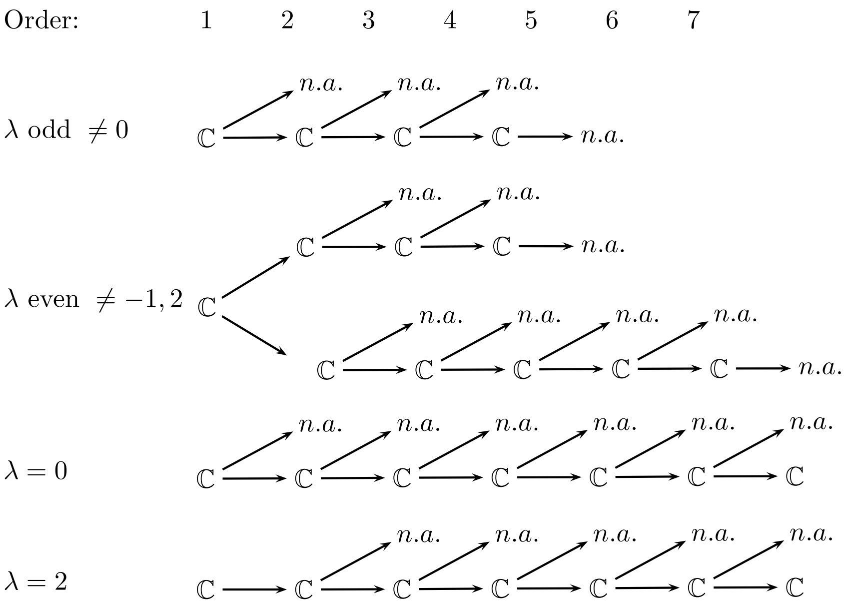
When we inject the solution in the next differential system, there could be multiplications by b, so we reduce the non homogeneous part modulo R_i. We find at the end

$$\frac{1}{7!}\partial_\theta^7 V_4 = \frac{57826741017348283}{893377392990720} b + \frac{25932696791821703}{100504956711456000} b^3$$

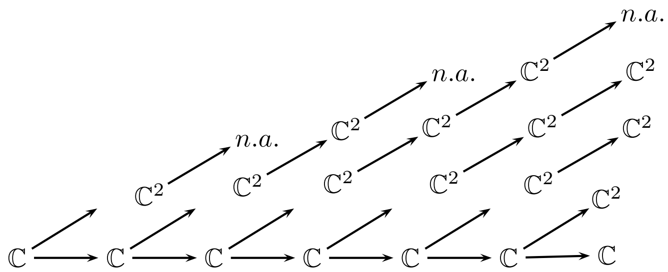
$$\frac{1}{7!}\partial_\theta^7 V_6 = \frac{1188281545485244987488668535038277777}{431797756299715943933989778480280} b + \frac{8633140425176867273801758981735627411}{528952251467152031319137478638343000000} b^3$$

There is only one condition at order 6 and it can be satisfied. At order 7, we have again two constraints, and this time we do not have a free parameter to fix anymore. These two conditions cannot be simultaneously satisfied. □

Classification of Galois groups



$\lambda = -1$



The graph is built using the following rules: At each order k, if there is branching, the branch choice depend on some condition on the $k+1$-th derivatives or lower. The branch noted *n.a.* correspond to non abelian Galois groups. The tree is infinite for eigenvalues $-1, 0, 2$, but at the end, only the lower branch correspond to integrable potentials at all orders (and these are truly integrable).

Remark. - These expansions of the potentials are unique and allow for each given potential to precisely compute the order at which it is integrable (here in the case of the eigenvalues $5, 9, 14, 20$). This classification is then only proved for the eigenvalues $-1, 0, 2, 5, 9, 14, 20$, and for bigger eigenvalues, this corresponds conjecture 2. But the algorithm we used can compute the same for an arbitrary fixed eigenvalue (belonging to $\{(n-1)(n+2)/2, \quad n \in \mathbb{N}\}$). The proof of conjecture 2 is then possible for many eigenvalues, even if the computation cost is great. Numerical experiments suggest that the complexity is subexponential $\exp\sqrt{\Lambda}$, where Λ is the eigenvalue bound (see Definition 11), and so is still tractable. This leads to a practical way to study in all generality bounded eigenvalue problems, at least in the case where the bounding is effective (known explicitly).

3.7 Other type of potentials

We have here only written about non degenerate Darboux points. Now we will take a look at the degenerate case.

Theorem 15. *Let V be a rational homogeneous potential on $\mathcal{C}$ of degree -1 with $V = \frac{1}{r}U(\theta)$ in polar coordinates. We assume there exists θ_0 such that*

$$U(\theta_0) = U'(\theta_0) = 0$$

If V has a first integral I, rational on $\mathbb{C}^2 \times \mathcal{C}$ and independent almost everywhere with H, then $U''(\theta_0) = 0$. Conversely, if $U''(\theta_0) = 0$, then the identity component of the Galois group of variational equation near θ_0 is abelian at any order.

Here we add the restriction that V is rational on $\mathcal{C}$ instead of meromorphic. This is due to the fact that the variational equation (see the proof below) is not Fuchsian, and thus the Galois group over meromorphic functions could be different from the Galois group over rational functions.

Proof. The first order variational equation is the following

$$\ddot{X}_1 = 0 \qquad \ddot{X}_2 = \frac{U''(\theta_0)}{t^3} X_2 \tag{3.22}$$

Let M be an open set of $\mathbb{C}^2 \times \mathcal{C}$, containing the orbit

$$\Gamma = \{q = (t\cos\theta_0, t\sin\theta_0),\ r = t,\ p = (\cos\theta_0, \sin\theta_0),\ t \in \mathbb{C}^*\}$$

and such that the Hamiltonian is holomorphic on M. To this orbit Γ, we add singular points $t = 0, \infty$, noting it $\bar{\Gamma}$. We now use Theorem 2. of Morales-Ramis-Simó [54] in its version with

$\bar{\Gamma}$. As said in their article, this Theorem is still valid when adding singular points to the orbit Γ, and then considering the differential Galois group over the meromorphic functions on $\bar{\Gamma}$ (see also Morales-Ramis [51] p 73). If the potential V is rationally integrable (and thus meromorphic on a neighbourhood of $\bar{\Gamma}$, the variational equation (3.22) should have a Galois group with an abelian identity component over the base field of meromorphic functions on $\bar{\Gamma}$. As $\bar{\Gamma} \simeq \bar{\mathbb{C}}$, this base field is the rational functions $\mathbb{C}(t)$.

Let us now compute the Galois group of equation (3.22). The first equation is clearly integrable. Assume now that $U''(\theta_0) \neq 0$. For the second one, we make a linear variable change and this gives

$$\ddot{y} = \frac{1}{t^3} y \tag{3.23}$$

Using the Kovacic algorithm, we find that the Galois group of this equation is $SL_2(\mathbb{C})$, and thus connected and non abelian. So the only possibility left is $U''(\theta_0) = 0$, for which equation (3.22) has a Galois group equal to $\{Id\}$ (thus abelian).

Now assume the reverse, that $U''(\theta_0) = 0$. The first order variational equation is written

$$\ddot{X}_1 = 0 \qquad \ddot{X}_2 = 0$$

We already know that Morales-Ramis-Simó integrability condition is satisfied at order 1, and we know want to test it at any order.

Lemma 18. *The algebra $\mathcal{A} = \mathbb{C}[t, \frac{1}{t}, \ln t]$ is stable by integration.*

Proof. We consider $f \in \mathbb{C}[t, \frac{1}{t}, \ln t]$ and we write it as a linear combination of terms of the type

$$t^n \ln(t)^m \quad n \in \mathbb{Z},\ m \in \mathbb{N}$$

If $n \geq 0$, then we use integration by parts to decrease m until 0. If $n < 0$, We use integration by part to increase n up to $n = -1$. We then have the formula

$$\int \frac{1}{t} \ln(t)^m dt = \frac{1}{m+1} \ln(t)^{m+1}$$

Then all functions in $\mathbb{C}[t, \frac{1}{t}, \ln t]$ have a primitive in $\mathbb{C}[t, \frac{1}{t}, \ln t]$. □

We now use this Lemma, remarking the following phenomenon. The solutions of higher variational equation are in fact solutions of non homogeneous linear differential equations and the non homogeneous terms are produced only using products of lower order solutions and functions t^{-k}. So the solutions always live in some algebra in which we take recursively integrations. So we apply the method of variation of constants to find the solutions. Moreover, the Wronskian of $\ddot{X}_2 = 0$ is equal to 1 (and also for the higher variational equations matrices), then we never have to divide. So all solutions live in the algebra $\mathcal{A}$ which is stable by integration. Then the Picard Vessiot field is

$$K_i = \mathbb{C}(t) \text{ or } \mathbb{C}(t, \ln\ t) \quad G_i = id \text{ or } \mathbb{C}$$

The Galois group is in both cases abelian at any order. □

Remark 8. *We were only able to analyze rational first integrals. Here the variational equation* (3.22) *is not Fuchsian at* 0*. This condition is a hypothesis of Combot Lemma 3 in [21] which proves that the Galois groups over the meromorphic functions and the Galois group over the rational functions are equal. Thus, we cannot use this Lemma, and we need to use Morales-Ramis-Simó Theorem over $\bar{\Gamma}$. So only the rational first integrals can be analyzed. To have a "reasonable definition of integrability", we than need to assume that V is rational, as if the potential V itself is not meromorphic for $r = 0$, for example when U is not a rational function in* $\exp i\theta$*, then the Hamiltonian (the only first integral we know in advance) would not be rational and then excluded from this analysis.*

This type of proof appears to be very general. Indeed, if some potential appears to be integrable at all order near a particular solution, without known first integral, the Picard Vessiot field is often not growing. If the coefficients of the potential are not well adjusted to avoid creating further monodromy, it is probably because it is not possible. Looking at the non linear version of variational equation in [54] p 860 (and also here section 3.4.2), we see that the solutions of higher variational equations are in fact solutions of non homogeneous linear differential equations and the non homogeneous terms are produced only using products. As such equation can be solved using the method of variation of parameters, the solutions always live in some algebra in which we take recursively integrations. For homogeneous potentials of degree -1 and non degenerate Darboux points, the algebra is given by the following process

$$\mathcal{A}_0 = \mathbb{C}\left[t, \frac{1}{t^2-1}\right], \qquad \mathcal{A}_{i+1} = \int \mathcal{A}_i dt$$

where $\int \mathcal{A}_i dt \supset \mathcal{A}_i$ is the algebra generated by all integrations of functions in A_i. We have then in particular that the Picard Vessiot of (VE_i) (in the case of all the eigenvalues belong to Morales Ramis table) is always contained in the fraction field of A_{i+1}. These algebras contains in particular all the polylogarithms functions that give integrability constraints, but not only.

3.8 Conclusion

Using this analysis, all bounded eigenvalue problems can be algorithmically studied. The two conjectures represents the last open questions about homogeneous potentials in the plane of degree -1. They could in principle be proved using the D-finiteness property of the functions P_n, Q_n (the fact that they satisfy linear polynomial recurrence and differential equations), but in practice direct computation seems to be way out of reach for the moment. No counter example have been found to conjectures 1, 2, and numerical computations strongly suggest that the integrability constraint at order 5 and 7 are never compatible. With these theorems we see that very often, the difficulty is not the number of parameters, but specific family of potentials (with even as low as 1 parameters) which exhibit the non eigenvalue bounding phenomenon.

Some examples of potentials integrable at all order near all Darboux points
In the case of non degenerate Darboux points, if we admit conjecture 2, it will not be possible to find non integrable potentials which are integrable all orders. We then need to find homogeneous potentials either having no Darboux points at all, either only multiple degenerate Darboux points (the second derivative should vanish). The functions $U(\theta) = F\left(e^{i\theta}\right)$ with

$$F(z) = h(z^n) \quad h \text{ Moebius transformation, } n \in \mathbb{N}^*$$

$$F(z) = f(z^n) \text{ with } f(z) = \int \frac{az^i}{(z-\alpha)^j} dz \;\; 0 \leq i \leq j-2, \; n \in \mathbb{N}^* \; \alpha \in \mathbb{C}^*$$

have no critical points. The functions $U(\theta) = F\left(e^{i\theta}\right)$ with

$$F(z) = h((z^n - \alpha)^m) - h(0) \quad m \geq 3, \; n \in \mathbb{N}^* \; \alpha \in \mathbb{C}^*$$

with h a Moebius transformation, have only degenerate Darboux points satisfying the integrability constraint.

These examples show that there are still open questions about integrability, but the difficulties do not rely on Morales Ramis theory but on the search of Darboux points. Potentials without non degenerate Darboux points are not common, but they still exist and a complete classification of them seems to be difficult.

Chapter 4

Third order variational equation for non generic cases

4.1 Introduction

In this chapter, we will be interested in non-integrability proofs of meromorphic homogeneous potentials of degree -1 in the plane, and in particular in non-generic cases. Writing our potential V in polar coordinates, and making the Fourier expansion in the angle gives us

$$V(r,\theta) = r^{-1} \sum_{k=-\infty}^{\infty} a_k e^{ik\theta}. \tag{4.1}$$

This type of potential covers many physical problems in celestial mechanics and n-body problems, in particular the anisotropic Kepler problem, the isosceles 3-body problem, the colinear 3-body problem, the symmetric 4-body problem and so on. Moreover, for such a potential there are strong integrability conditions, thanks to the Morales-Ramis theory [51] and to a very effective criterion of Yoshida [74]. Still, for such a general potential, this criterion will not be sufficient. This is not particularly because this class of potentials is large, but because there are non-generic, very resistant cases inside. For example, if we want to study the integrability of $V(r,\theta) = r^{-1}h(\exp(i\theta))$ with a polynomial h, we have a priori a potential with $\deg h + 1$ complex parameters, and Yoshida's integrability criterion will restrict this family to a family with $\deg h - 1$ integer parameters. Still one would like to have a finite list of possible integrable potentials, so as to be able to check the existence of first integrals one by one. Here we will present a stronger criterion in Theorems 17 and 18 which is able to deal with such families, and which therefore is capable to settle any integrability question on finite dimensional families of type (4.1). As an application of our method, we will apply this criterion in the case $V(r,\theta) = r^{-1}h(\exp(i\theta))$ with $h \in \mathbb{C}[z]$, $\deg h \leq 3$. To do precise statements, let us now begin with some definitions concerning homogeneous potentials and integrability.

Definition 13. *We consider the algebraic variety $\mathcal{S} = \{(q_1,q_2,r) \in \mathbb{C}^3,\ \ r^2 = q_1^2 + q_2^2\}$ and the derivations for a function f on $\mathcal{S}$*

$$\frac{\partial f}{\partial q_1} = \partial_1 f + r^{-1} q_1 \partial_3 f, \quad \frac{\partial f}{\partial q_2} = \partial_2 f + r^{-1} q_2 \partial_3 f$$

where ∂_i is the derivative according to the i-th variable (the variables of f are q_1, q_2, r in this order). This defines a symplectic form on $\mathbb{C}^2 \times \mathcal{S}$ on which we consider a Hamiltonian H of the form

$$H(p_1,p_2,q_1,q_2,r) = \frac{1}{2}(p_1^2 + p_2^2) - V(q_1,q_2,r)$$

with the associated system of differential equations

$$\dot{r} = r^{-1}(q_1\dot{q}_1 + q_2\dot{q}_2), \quad \dot{q}_i = \frac{\partial}{\partial p_i} H, \quad \dot{p}_i = -\frac{\partial}{\partial q_i} H, \quad i = 1,2. \tag{4.2}$$

The potential V is assumed to be meromorphic on $\mathcal{S}$ and to have the following form in polar coordinates:

$$V(r,\theta) = \frac{1}{r} U(\theta), \quad r\cos\theta = q_1, \quad r\sin\theta = q_2$$

This implies that V is homogeneous of degree -1. We say that I is a meromorphic first integral of H, if I is a meromorphic function on $\mathbb{C}^2 \times \mathcal{S}$ such that

$$\dot{I} = \{H, I\} = \sum_{i=1}^{2} \left(\frac{\partial}{\partial p_i} H \frac{\partial}{\partial q_i} I - \frac{\partial}{\partial q_i} H \frac{\partial}{\partial p_i} I \right) = 0.$$

Obviously, the Hamiltonian H itself is a first integral. We will say that V is meromorphically integrable if it possesses an additional meromorphic first integral which is independent almost everywhere from H.

Definition 14. *We call* $c = (c_1, c_2, c_3) \in \mathcal{S}$ *a Darboux point of* V *if*

$$\frac{\partial}{\partial q_1} V(c) = \alpha c_1 \quad \textit{and} \quad \frac{\partial}{\partial q_2} V(c) = \alpha c_2 \tag{4.3}$$

where $\alpha \in \mathbb{C}$ *is called the multiplier. Because* V *has singularity at* $c_3 = 0$*, we will* ***always*** *assume that* $c_3 \neq 0$*. Because of homogeneity, we can always choose* $\alpha = 0$ *or* $\alpha = -1$*. We say that* c *is non-degenerate if* $\alpha \neq 0$*. To the Darboux point* c *we associate a homothetic orbit given by*

$$r(t) = c_3\phi(t), \quad q_i(t) = c_i\phi(t), \quad p_i(t) = c_i\dot{\phi}(t) \qquad (i = 1, 2), \tag{4.4}$$

with ϕ *satisfying the following differential equation*

$$\frac{1}{2}\dot{\phi}(t)^2 = -\frac{\alpha}{\phi(t)} + E, \quad E \in \mathbb{C}.$$

In the following, we will often omit the last component of a Darboux point $c \in \mathcal{S}$ as it is defined up to a sign (and the choice of sign does not matter) by the two first components.

Definition 15. *The first order variational equation of* H *near a homothetic orbit is given by*

$$\ddot{X}(t) = \frac{1}{\phi(t)^3} \nabla^2 V(c) X(t)$$

where $\nabla^2 V(c)$ *is the Hessian of* V *(according to derivations in* q*). After diagonalization (if possible) and the change of variable* $\phi(t) \longrightarrow t$*, the equation simplifies to*

$$2t^2(Et+1)\ddot{X}_i - t\dot{X}_i = \lambda_i X_i,$$

where the λ_i *are the eigenvalues of the Hessian of* V *evaluated at the Darboux point* c*, i.e.,* $\lambda_i \in \operatorname{Sp}\left(\nabla^2 V(c)\right)$.

Theorem 16. *(Morales, Ramis, Simó Yoshida [74, 54, 51]) If* V *is meromorphically integrable, then the neutral component of the Galois group of the variational equation near a homothetic orbit with* $E \neq 0$ *is abelian at all orders. If we fix the multiplier of the associated Darboux point to* -1*, the Galois group of the first order variational equation has an abelian neutral component if and only if*

$$\operatorname{Sp}\left(\nabla^2 V(c)\right) \subset \left\{\tfrac{1}{2}(k-1)(k+2) : \ k \in \mathbb{N}\right\}.$$

If the multiplier of the Darboux point is 0*, the Galois group of the first order variational equation has an abelian neutral component if and only if*

$$\operatorname{Sp}\left(\nabla^2 V(c)\right) \subset \{0\}.$$

In fact, this is not exactly the same statement as the original theorem because we allow r to appear in the potential and in the first integrals.

Proof. Let $\Gamma \subset \mathbb{C}^2 \times \mathcal{S}$ denote the curve defined by equation (4.4) without the singular point $(q_1, q_2, r) = (0, 0, 0)$, and M an open neighbourhood of Γ in $\mathbb{C}^2 \times \mathcal{S}$ such that H is holomorphic on M. The Hamiltonian H is then well defined and holomorphic on a symplectic manifold M and the additional first integral is meromorphic on M. Hence, using the main theorem of [54], the neutral component of the Galois group of the variational equation near Γ is abelian at all orders over the base field of meromorphic functions on Γ. The variational equation is a hypergeometric equation. In [37], Kimura classifies Galois groups of hypergeometric equations over the base field $\mathbb{C}(t)$. We can use this classification as the Galois group over the base field $\mathbb{C}(t)$ is the same as over the base field of meromorphic functions because the hypergeometric equation is a Fuchsian equation (see page 73 of [51]). This produces the condition on the spectrum of $\nabla^2 V(c)$. The case of a degenerate Darboux point leads to the variational equation

$$\ddot{X} = \lambda t^{-3} X$$

which is a Bessel equation (after a change of variables). Its Galois group over the field of meromorphic functions in t has not an abelian identity component except if $\lambda = 0$. $\square$

Note that in the case of a degenerate Darboux point, we explicitly need that the first integral is meromorphic including $r = 0$, as the variational equation is not regular singular at this point. The integrability condition for a non-degenerate Darboux point also holds for a potential meromorphic only on $\mathcal{S}^* = \mathcal{S} \setminus \{r = 0\}$ and meromorphic first integrals on $\mathbb{C}^2 \times \mathcal{S}^*$.

4.2 Main Results

In this section, we are going to state the main theorems of this article. The remaining parts of this paper are dedicated to their proofs.

Theorem 17. *Let V be a homogeneous potential of degree -1 in the plane. We assume that $c = (1, 0)$ is a Darboux point of V with multiplier -1. If the variational equation is integrable at order 3, then the following conditions are fulfilled*

$$\operatorname{Sp}\left(\nabla^2 V(c)\right) = \left\{2, \tfrac{1}{2}(p-1)(p+2)\right\} \text{ for some } p \in \mathbb{N}.$$

If p is even then

$$\left(\frac{\partial^3 V}{\partial q_1 \partial q_2^2}\right)^2 f_1(p) + \left(\frac{\partial^3 V}{\partial q_2^3}\right)^2 f_2(p) + \left(\frac{\partial^4 V}{\partial q_2^4}\right) f_3(p) = 0,$$

and if p is odd then

$$\frac{\partial^3 V}{\partial q_2^3} = 0 \quad \text{and} \quad \left(\frac{\partial^3 V}{\partial q_1 \partial q_2^2}\right)^2 f_1(p) + \left(\frac{\partial^4 V}{\partial q_2^4}\right) f_3(p) = 0,$$

where the functions f_1, f_2, f_3 satisfy explicit P-finite recurrences, i.e., linear recurrences with polynomial coefficients.

This theorem is a generalization of the criterion given by Yoshida for homogeneous potentials in the case of degree -1 and dimension 2. A similar theorem could be proven in higher dimensions, but the main problem is that Theorem 17 is almost inapplicable in this form. In most cases, it is necessary to study more closely the expression of the functions $f_1(p), f_2(p), f_3(p)$ to apply it, and for the moment, because of limitations of computing power, it seems only possible to do in dimension 2 (for which the computations are already tedious).

Theorem 18. *The functions $f_1(2n), f_2(2n), f_3(2n)$ can be written as*

$$\begin{aligned} f_1(2n) &= \epsilon_1(n)\left(\frac{1511011}{67108864n^2} - \frac{1511011}{134217728n^3} + \frac{31731231}{4294967296n^4}\right) \\ f_2(2n) &= \epsilon_2(n)\left(\frac{22665165}{1073741824n^4} - \frac{22665165}{1073741824n^5} + \frac{298125}{4194304n^6}\right) \\ f_3(2n) &= \epsilon_3(n)\left(-\frac{1740684681}{68719476736n^2} + \frac{1740684681}{137438953472n^3} - \frac{2400813907}{68719476736n^4}\right) \end{aligned} \tag{4.5}$$

with

$$|\epsilon_i(n) - 1| \leq 10^{-5} \quad \forall n \geq 100.$$

With this, we can apply Theorem 17 to some concrete examples:

Theorem 19. *Let V be a potential in the plane expressed in polar coordinates by*

$$V(r,\theta) = r^{-1}\left(a + be^{i\theta} + ce^{2i\theta} + de^{3i\theta}\right). \tag{4.6}$$

If V is meromorphically integrable, then V belongs to one of the following families

$$\begin{aligned} &V = r^{-1}a, && V = r^{-1}\big(a + be^{i\theta}\big), && V = r^{-1}\big(ae^{i\theta} + be^{3i\theta}\big),\\ &V = r^{-1}\big(a + be^{2i\theta}\big), && V = r^{-1}\big(a + be^{3i\theta}\big), && V = r^{-1}\big(a + be^{i\theta}\big)^3, \end{aligned} \tag{4.7}$$

with $a, b \in \mathbb{C}$.

The first three families have already known additional first integrals [31], polynomial of degree 1 or 2 in p. The status of the last three families is unknown. This is not due to an incomplete application of the Morales-Ramis Theorem, but linked to the fact that either they do not possess any Darboux points, or in the last case the only Darboux point is very degenerate and therefore the Morales-Ramis Theorem gives no integrability constraints at any order, as proven in [19].

In practical problems like Theorem 19, studying integrability only using the Morales-Ramis criterion is impossible because of two facts. First we need a Darboux point of our problem; if we do not have any, the only thing we can do is to try to find an additional first integral using the direct method of Hietarinta [31].

The second problem is the following scenario: inside the family of potentials given by Theorem 19, there exist submanifolds in the space of parameters for which the potential possesses only one Darboux point and the eigenvalue at this Darboux point can be arbitrarily high. In this case, the higher variational method is required. But the constraint at order 2 does not give sufficient conditions to conclude, and it is necessary to go to order 3.

But the expression of this constraint cannot be written explicitly for all possible eigenvalues, only for a finite number of them. To apply this third-order criterion, we derive P-finite recurrences and asymptotic expansions with error control in Theorem 18. This allows us to prove that the integrability condition is not fulfilled. The proof of Theorem 19 therefore will be split into two parts:

1. The first part consists in constructing a manifold M in the space of the parameters a, b, c, d such that if the eigenvalues for all Darboux points are real, then the parameters belong to M. Then we produce a decomposition $M = M_1 \cup \dots M_k$ and study each manifold separately. For some of them, the corresponding potentials possess sufficiently many Darboux points to give a strong enough condition for integrability only using the Morales-Ramis criterion at order 1 (there could exist some resistant cases for which a higher variational equation is needed but without the phenomenon of arbitrary high eigenvalues like in [46]). But for specific cases, this phenomenon occurs. It has already been noticed by Maciejweski in [44] who lets this specific case open.

2. The second part will be devoted to these specific manifolds M_i where the Morales-Ramis criterion at order 1 is almost powerless. We use Theorems 17 and 18 to solve these hard cases.

In [15], the authors deal with a similar difficulty with the spring pendulum for which there is a discrete infinite set of parameters for which there are no obstructions to integrability at order 2. They also study third order variational equations, but then use analytic tools to study a sequence of monodromy elements, and finally prove that this sequence never vanishes. Thanks to our explicit expression via P-finite recurrences, such a problem can be analysed more systematically here.

4.3 Eigenvalue Bounding

Definition 16. *We will denote*

$$\mathcal{M} = \left\{ V(r,\theta) = r^{-1}U(\theta) \text{ with } U \text{ meromorphic and } 2\pi\text{-periodic} \right\}.$$

Let $V \in \mathcal{M}$. We denote by $d(V)$ the set of Darboux points c of V with multiplier -1 and $c_3 \neq 0$. For $c \in d(V)$ we have $\mathrm{Sp}\left(\nabla^2 V(c)\right) = \{2, \lambda\}$ and we denote

$$\Lambda(c) = \begin{cases} \lambda & \text{if } \lambda \in \mathbb{R} \\ -\infty & \text{otherwise} \end{cases}.$$

Definition 17. *We consider a subset $E \subset \mathcal{M}$ and define*

$$\Lambda(E) = \sup_{V \in E,\, d(V) \neq \varnothing} \inf_{c \in d(V)} \Lambda(c).$$

We say that the problem of finding all meromorphically integrable potentials in E is a bounded eigenvalue problem if $\Lambda(E) < \infty$.

Remark 9. *We have $\Lambda(\mathcal{M}) = \infty$ because of the following family*

$$V(r,\theta) = r^{-1}\left((1+a) - 2ae^{i\theta} + ae^{2i\theta}\right), \quad a \in \mathbb{R},$$

for which only one Darboux point $c = (1,0)$ exists; the corresponding eigenvalue is $\lambda = 2a - 1$. This proves that the family of potentials considered in Theorem 19 is an unbounded eigenvalue problem.

Lemma 19. *For a potential $V \in \mathcal{M}$ the Darboux points c such that $c_3 \neq 0$ can be written as $c = (c_1, c_2) = (r_0\cos(\theta_0), r_0\sin(\theta_0))$ with θ_0 being a critical point of U. The Darboux point c is non-degenerate if and only if $U(\theta_0) \neq 0$ and in this case, the eigenvalues of the Hessian of V, evaluated at c, are*

$$\mathrm{Sp}\left(\nabla^2 V(c)\right) = \left\{ 2, \frac{U''(\theta_0)}{U(\theta_0)} - 1 \right\}.$$

if we choose the multiplier of c to be -1.

Proof. For $V = r^{-1}U(\theta)$ the conditions (4.3) that c is a Darboux point are:

$$\begin{aligned} r_0^{-3}\left(-c_1 U(\theta_0) - c_2 U'(\theta_0)\right) &= \alpha c_1, \\ r_0^{-3}\left(-c_2 U(\theta_0) + c_1 U'(\theta_0)\right) &= \alpha c_2. \end{aligned}$$

Assuming $c_3 \neq 0$, it follows that $U(\theta_0) = -\alpha r_0^3$ and $U'(\theta_0) = 0$, which means that θ_0 is a critical point of U. Since $c_3 \neq 0$ implies that $r_0 \neq 0$, we see that the case $\alpha = 0$ (degenerate Darboux point) is equivalent to $U(\theta_0) = 0$. Setting $\alpha = -1$ and $U'(\theta_0) = 0$ we get the Hessian matrix

$$\nabla^2 V(c) = \frac{1}{r_0^5} \begin{pmatrix} (2c_1^2 - c_2^2)U(\theta_0) + c_2^2 U''(\theta_0) & c_1 c_2 (3\,U(\theta_0) - U''(\theta_0)) \\ c_1 c_2 (3\,U(\theta_0) - U''(\theta_0)) & (2c_2^2 - c_1^2)U(\theta_0) + c_1^2 U''(\theta_0) \end{pmatrix}$$

whose eigenvalues are exactly those claimed above (using $U(\theta_0) = r_0^3$). ☐

Recall that the potentials given by (4.6) are $V(r,\theta) = r^{-1}U(\theta)$ with $U(\theta) = a + be^{i\theta} + ce^{2i\theta} + de^{3i\theta}$. We now assume that V possesses at least one non-degenerate Darboux point c with $c_3 \neq 0$. After rotation, we can always assume that $c = (1,0)$ is a Darboux point. As shown in Lemma 19, it corresponds to a critical point for $\theta = 0$. Moreover, because this Darboux point is non-degenerate, we know that $U(0) \neq 0$. Then by dilatation, we can also assume that $U(0) = 1$ and get the following equations

$$U(0) = a + b + c + d = 1,$$
$$U'(0) = i(b + 2c + 3d) = 0.$$

Solving these equations for c and d, yields the expression

$$V_{a,b} = r^{-1}\left(a + be^{i\theta} + (3 - 3a - 2b)e^{2i\theta} + (2a + b - 2)e^{3i\theta}\right)$$

for the potentials where $a, b \in \mathbb{C}$.

Theorem 20. *If $V_{a,b}$ is meromorphically integrable, then it belongs to one of the following families*

$$E_1 = r^{-1}\left(-\frac{1}{3}b + 1 + be^{i\theta} - be^{2i\theta} + \frac{1}{3}be^{3i\theta}\right),$$
$$E_2 = r^{-1}\left(-\frac{1}{6}k(k+1)e^{3i\theta} + \frac{1}{4}k(k+1)e^{2i\theta} - \frac{1}{12}k^2 - \frac{1}{12}k + 1\right),$$
$$E_3 = r^{-1}\left(-\frac{1}{4}k(k+1)e^{2i\theta} + \frac{1}{2}k(k+1)e^{i\theta} - \frac{1}{4}k^2 - \frac{1}{4}k + 1\right),$$
$$E_4 = r^{-1}\left(\frac{(s - 6\lambda_2)\lambda_2}{18(\lambda_1 + \lambda_2)}e^{3i\theta} - \frac{(3\lambda_1 + s - 3\lambda_2)\lambda_2}{6(\lambda_1 + \lambda_2)}e^{2i\theta} + \frac{(6\lambda_1 + s)\lambda_2}{6(\lambda_1 + \lambda_2)}e^{i\theta} + \frac{-9\lambda_1\lambda_2 - \lambda_2 s + 18\lambda_1 + 18\lambda_2 - 3\lambda_2^2}{18(\lambda_1 + \lambda_2)}\right)$$

where $b \in \mathbb{C}$ and $k \in \mathbb{N}$. The quantities arising in E_4 are

$$s^2 = 6\lambda_1^2\lambda_2 + 6\lambda_1\lambda_2^2 - 36\lambda_1\lambda_2,$$
$$\lambda_i = \frac{1}{2}(k_i - 1)(k_i + 2) + 1 \quad (i = 1, 2),$$

with $k_1 \in \mathbb{N} \setminus \{0, 3\}$ and $k_2 \in \mathbb{N}^*$.

Proof. For all non-degenerate Darboux points $c = (\gamma\cos(\theta_0), \gamma\sin(\theta_0))$ the corresponding eigenvalue λ satisfies

$$U''(\theta_0) - (\lambda + 1)U(\theta_0) = 0 \quad \text{and} \quad U'(\theta_0) = 0 \tag{4.8}$$

(note that this condition is also satisfied if c is degenerate). We write $U(\theta) = h_{a,b}(\exp(i\theta))$, $U'(\theta) = izh'_{a,b}(\exp(i\theta))$, and $U''(\theta) = \tilde{h}_{a,b}(\exp(i\theta))$ with

$$h_{a,b}(z) = a + bz + (3 - 3a - 2b)z^2 + (2a + b - 2)z^3,$$
$$\tilde{h}_{a,b}(z) = -bz - 4(3 - 3a - 2b)z^2 - 9(2a + b - 2)z^3.$$

So to find the eigenvalues of all Darboux points, one just needs to compute the following resultant which corresponds to the conditions (4.8):

$$\begin{aligned} P_{a,b}(\lambda) &= \operatorname{res}_z\left(\tilde{h}_{a,b}(z) - (\lambda+1)h_{a,b}(z),\, h'_{a,b}(z)\right) \\ &= (2a + b - 2)(6a + 2b - 6 + (\lambda + 1))(-18ab^2 - 6b^3 + 18b^2 + (\lambda + 1) \\ &\quad \times (108a^3 + 108a^2b - 216a^2 + 36ab^2 + 108a - 108ab - 9b^2 + 4b^3)) \end{aligned}$$

All the roots of $P_{a,b}(\lambda)$ correspond to an eigenvalue of some Darboux point, except possibly in those cases (a, b) where $P_{a,b}$ vanishes as a polynomial in λ or in the case where $h'_{a,b}(z)$ has the root 0.

Let us begin with the special cases. We compute the points $(a, b) \in \mathbb{C}^2$ for which $P_{a,b} = 0$ in $\mathbb{C}[\lambda]$. We find that it is the zero set of the ideal $\langle 2a + b - 2 \rangle \cap \langle a, b \rangle$. Moreover, the polynomial $h'_{a,b}(z)$ has a zero root if and only if $b = 0$. So, all the specific cases belong to the zero set of $\langle 2a + b - 2 \rangle \cap \langle b \rangle$. First, for $b = 0$ we find

$$\begin{aligned} Q_1 &= \operatorname{res}_z(\tilde{h}_{a,0}(z) - (\lambda + 1)h_{a,0}(z),\, h'_{a,0}(z)/z, z) \\ &= 216(a - 1)^3(6a - 6 + (\lambda + 1)), \end{aligned}$$

and second, for $b = 2 - 2a$ we get

$$\begin{aligned} Q_2 &= \operatorname{res}_z(\tilde{h}_{a,2-2a}(z) - (\lambda + 1)h_{a,2-2a}(z),\, h'_{a,2-2a}(z), z) \\ &= -4(a - 1)^2(2a - 2 + (\lambda + 1)). \end{aligned}$$

As we know that the eigenvalues should be of the form $\frac{1}{2}(k - 1)(k + 2)$, $k \in \mathbb{N}$, we obtain the potentials E_2 and E_3 from these two cases.

Now for the generic case, we express a and b depending on the roots of $P_{a,b}(\lambda)$ and obtain the expression E_4. Since it is not valid for $k_1 = k_2 = 0$, we study this case separately and find the condition $a = -\frac{1}{3}b + 1$, which gives E_1. Note that fixing $\lambda_1 = 0$ in E_4 yields the potential E_2, whereas $\lambda_1 = 6$ results in E_3. The case $k_2 = 0$ produces $V = r^{-1}$ which already belongs to E_1. □

Corollary 3. *With the same notation as in Theorem 20, we have* $\Lambda(E_1) = -1$ *and* $\Lambda(E_2) = \Lambda(E_3) = \Lambda(E_4) = \infty$.

Remark 10. *The types of E_2, E_3 and E_4 differ fundamentally although they are all unbounded eigenvalue problems. This is because the dimension of E_4 is 2 and the dimension of E_2 and E_3 is only 1. Because of that, we could call E_4 a doubly unbounded eigenvalue problem because it possesses two Darboux points whose eigenvalues can be independently arbitrarily high. Because of that, we will need to apply a third order integrability criterion simultaneously at the two Darboux points. The potential E_1 has only one Darboux point with eigenvalue -1. This eigenvalue belongs to the Morales-Ramis table and so higher variational methods will be required, but only for this fixed eigenvalue (which is much easier).*

In the parameter space, we get 4 algebraic manifolds. For E_2, E_3, and E_4, a tedious treatment with higher variational equations is required. For E_1 we will be able to check integrability easily with Theorem 17. A similar procedure could be applied to any set of homogeneous potentials depending rationally on some parameters. Here computing power is the main limitation; in particular, because for typical problems, the number of parameters is much smaller than the number of roots which requires resultant computations and prime ideal decompositions. One should note that we have deliberately chosen a set of potentials (4.6) which is particularly difficult to treat. For most common problems (outside the general complete classification), these unbounded eigenvalue manifolds have small dimension (1 in the case found by [44]) or even inexistent like in [46] or [55].

4.4 Higher Order Variational Methods

We will first recall some properties of the solutions of the first order variational equations. After diagonalisation and in the integrable case, the equation is the following (after fixing the energy $E = 1$)

$$2t^2(1 + t)y''(t) - ty'(t) - \tfrac{1}{2}(n - 1)(n + 2)y(t) = 0 \qquad (n \in \mathbb{N}). \tag{4.9}$$

After the change of variables $t \longrightarrow (t^2 - 1)^{-1}$, this equation becomes

$$(t^2 - 1)y''(t) + 4ty'(t) - (n - 1)(n + 2)y(t) = 0 \qquad (n \in \mathbb{N}). \tag{4.10}$$

A basis of solutions is given by (P_n, Q_n) where P_n are polynomials in t (for $n \geq 1$) and the functions Q_n are

$$Q_n(t) = P_n(t) \int \frac{1}{(t^2-1)^2 P_n(t)^2} dt.$$

The functions Q_n are multivalued except for $n = 0$ which will be a special case. Indeed, the Galois group of (4.9) in this case is *Id* instead of $\mathbb{C}$.

The polynomials P_n can be computed using the Rodrigues type formula

$$P_n(t) = \frac{1}{t^2-1} \frac{\partial^{n-1}}{\partial t^{n-1}} (t^2-1)^n \qquad (n \geq 1) \tag{4.11}$$

which also gives a normalisation for their leading coefficient. The functions Q_n can be written as

$$Q_n(t) = \epsilon_n P_n(t) \operatorname{arctanh}\left(\frac{1}{t}\right) + \frac{W_n(t)}{t^2-1} \qquad (n \geq 1) \tag{4.12}$$

where W_n are polynomials given by

$$\begin{aligned} W_{2k}(t) = \frac{(-1)^k(t^2-1)}{2^{4k}} \Bigg(& \frac{\pi\ {}_2F_1\big(\frac{1}{2}-k, k+1, \frac{1}{2}, t^2\big)}{\Gamma\big(k+\frac{1}{2}\big)^2} + \\ & \frac{2kt(2k+1)\operatorname{arctanh}(t)\ {}_2F_1\big(1-k, k+\frac{3}{2}, \frac{3}{2}, t^2\big)}{(k!)^2} \Bigg), \\ W_{2k+1}(t) = \frac{(-1)^k(t^2-1)}{2^{4k+2}} \Bigg(& \frac{\pi t(k+1)(2k+1) {}_2F_1(\frac{1}{2}-k, k+2, \frac{3}{2}, t^2)}{\Gamma(k+\frac{3}{2})^2} - \\ & \frac{2\operatorname{arctanh}(t)\ {}_2F_1(-k, k+\frac{3}{2}, \frac{1}{2}, t^2)}{(k!)^2} \Bigg) \end{aligned}$$

and ϵ_n is a real sequence given by

$$\epsilon_n = \frac{n(n+1)}{4^n(n!)^2}.$$

Conventionally, we will take for $n = 0$:

$$P_0(t) = \frac{t}{t^2-1}, \qquad Q_0(t) = \frac{1}{t^2-1}.$$

Lemma 20. *The functions $P_n(t)$ and $\frac{1}{\epsilon_n}Q_n(t)$ satisfy the differential equation* (4.10) *and the three-term recurrence*

$$4n(n+1)(n+2)y_n(t) - 2t(n+2)(2n+3)y_{n+1}(t) + (n+3)y_{n+2}(t) = 0.$$

Proof. Given the explicit expressions (4.11) and (4.12) we can use holonomic closure properties to derive the differential equation resp. recurrence they satisfy. We first express (4.11) as

$$P_n(t) = \frac{(n-1)!}{2\pi i(t^2-1)} \oint \frac{(u^2-1)^n}{(u-t)^n} du$$

by Cauchy's differentiation formula. By the method of creative telescoping we obtain the differential equation and the recurrence (this calculation was carried out by the software package `HolonomicFunctions` [38, 40]). Similarly we can apply holonomic closure properties to the closed form expression (4.12). □

Lemma 21. *(proved in [20]) Let $F \in \mathbb{C}(z_1)\,[z_2]$ and $f(t) = F\big(t, \operatorname{arctanh}\big(\frac{1}{t}\big)\big)$. We consider the field extension*

$$K = \mathbb{C}\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right), \int f\,dt\right)$$

and the monodromy group $G = \sigma(K, \mathbb{C}(t))$. If G is abelian, then

$$\frac{\partial}{\partial \alpha} \underset{t=\infty}{Res}\; F\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right) + \alpha\right) = 0.$$

Proof. We will consider two paths, the "eight" path σ_1 around the singularities -1 and 1, and the path σ_2 around infinity. At infinity, the function $F\big(t, \operatorname{arctanh}\big(\frac{1}{t}\big) + \alpha\big)$ will have a series expansion of the kind

$$\int F\left(t, \operatorname{arctanh}\left(\frac{1}{t}\right) + \alpha\right) dt = \sum_{n=n_0}^{\infty} a_n(\alpha) t^n + r(\alpha) \ln t$$

because the function $\operatorname{arctanh}\big(\frac{1}{t}\big)$ has a regular point at infinity. Let us now consider the monodromy commutator

$$\sigma = \sigma_2^{-1} \sigma_1^{-\frac{\beta}{2i\pi}} \sigma_2 \sigma_1^{\frac{\beta}{2i\pi}} \qquad \beta \in 2i\pi\mathbb{Z}.$$

We have that $\sigma_1^{\frac{\beta}{2i\pi}}(f) = F\big(t, \operatorname{arctanh}\big(\frac{1}{t}\big) + \beta\big)$ and $\sigma_2(\ln t) = \ln t + 2i\pi$. We deduce that

$$\sigma(f) = f + r(\beta) - r(0).$$

This $r(\beta)$ corresponds to the residue of $F\big(t, \operatorname{arctanh}\big(\frac{1}{t}\big) + \beta\big)$ at infinity. If the monodromy is commutative, then the commutator σ should act trivially on f. This is the case only if $r(\beta) - r(0) = 0$ for all $\beta \in 2i\pi\mathbb{Z}$. The function r is a polynomial in β, so $r(\beta) - r(0) = 0$ for all $\beta \in \mathbb{C}$. From this the claim follows. □

In the following, we will also need to use the next lemma which is a kind of reciprocal version of Lemma 21.

Lemma 22. *(proved in [20]) We consider*

$$F(t) = \sum_{i=0}^{3} H_i(t) \operatorname{arctanh}\left(\frac{1}{t}\right)^i$$

with $H_0, \dots, H_3 \in \mathbb{C}[t]$. If the conditions of Lemma 21 are satisfied, then

- *If $\underset{t=\infty}{Res}\; F(t) = 0$, then $\int F\,dt \in \mathbb{C}\left[t, \operatorname{arctanh}\left(\frac{1}{t}\right)\right]$*
- *If $\underset{t=\infty}{Res}\; F(t) \neq 0$, then $\int F\,dt \in \mathbb{C}\left[t, \operatorname{arctanh}\left(\frac{1}{t}\right), \ln\left(t^2 - 1\right)\right]$*

Theorem 21. *Let V be a homogeneous potential of degree -1 in the plane. We assume that $c = (1, 0)$ is a Darboux point of V with multiplier -1. If the variational equation is integrable at order 2 then*

$$\operatorname{Sp}\left(\nabla^2 V(c)\right) = \left\{2, \tfrac{1}{2}(p-1)(p+2)\right\}, \quad p \in \mathbb{N},$$

and for odd p we have $\frac{\partial^3 V}{\partial q_2^3} = 0$.

This theorem is in fact a particular case of Theorem 2 in [20] for which the three indices i, j, k are equal.

Remark 11. *Because the constraint appears only for odd p, the variational equations of order 2 give no constraint for even p. Hence this is not sufficient for proving non-integrability for an unbounded manifold.*

4.5 Proof of Theorem 17

Proof. The variational equation at order 3 is given by

$$\begin{aligned}
\ddot{X}_1 &= \frac{2}{\phi^3}X_1 + \frac{1}{2}\frac{a}{\phi^4}Y_{1,1} - \frac{4b}{3\phi^5}Z^3\\
\ddot{X}_2 &= \frac{\lambda}{\phi^3}X_2 + \frac{a}{\phi^4}Y_{2,1} + \frac{b}{\phi^4}Y_{1,1} + \frac{c}{\phi^5}Z^3\\
\dot{Y}_{1,1} &= 2Y_{1,2}\\
\dot{Y}_{1,2} &= \frac{\lambda}{\phi^3}Y_{1,1} + \frac{b}{\phi^4}Z^3 + Y_{1,3}\\
\dot{Y}_{1,3} &= \frac{\lambda}{\phi^3}Y_{1,2} + \frac{b}{\phi^4}Z^2\dot{Z}\\
\dot{Y}_{2,1} &= Y_{2,2} + Y_{2,3}\\
\dot{Y}_{2,2} &= \frac{2}{\phi^3}Y_{2,1} - \frac{4b}{3\phi^5}Z^3 + Y_{2,4}\\
\dot{Y}_{2,3} &= \frac{\lambda}{\phi^3}Y_{2,1} + Y_{2,4}\\
\dot{Y}_{2,4} &= \frac{2}{\phi^3}Y_{2,3} - \frac{4b}{3\phi^5}Z^2\dot{Z} + \frac{\lambda}{\phi^3}Y_{2,2}\\
\ddot{Z} &= \frac{2}{\phi^3}Z
\end{aligned}$$

where $\lambda = \frac{1}{2}(n-1)(n+2)$. The coefficients a, b, c correspond to the following derivatives

$$a = \frac{\partial^3}{\partial q_1 \partial q_2^2}V(c), \quad b = \frac{1}{2}\frac{\partial^3}{\partial q_2^3}V(c), \quad c = \frac{1}{6}\frac{\partial^4}{\partial q_2^4}V(c),$$

and the others are given using the Euler relation for homogeneous functions. A complete procedure to build these equations is given by [5]. The functions $Y_{1,1}$ and $Y_{2,1}$ are solutions of a system of linear differential equations with an inhomogeneous term, and the homogeneous part is in fact a symmetric product of the first order variational equation. Here, we already put to zero terms that we think in advance they will not produce integrability constraints. As before, we use the change of variables $\phi(t) \longrightarrow (t^2-1)^{-1}$.

We choose $Z(t) = Q_n$ and compute the solution for X_2 of the above system. We first remark that X_2 is in the Picard-Vessiot field, so it is also the case for its derivative. We now perform integration by parts and see that one term is already in the Picard-Vessiot field, and the other is

$$\int 2(t^2-1)^2 \left(a^2 t P_n Q_n I_1 + 4b^2 P_n^2 Q_n I_2 + c(t^2-1)Q_n^4\right) dt \tag{4.13}$$

where

$$\begin{aligned}
I_1 &= \int \left(\frac{\int \left(\frac{t(t^2-1)^2 Q_n^3}{P_n} + \frac{I_3}{(t^2-1)^2 P_n^2}\right) dt}{t^2(t^2-1)^2} + \frac{\int \frac{I_3}{t^2(t^2-1)^2} dt}{(t^2-1)^2 P_n^2} \right) dt\\
I_2 &= \int \frac{\int \left((t^2-1)^2 Q_n^3 + \frac{2}{(t^2-1)^2 P_n^2}\int (t^2-1)^4 P_n Q_n^2 (P_n \dot{Q}_n - Q_n \dot{P}_n) dt\right) dt}{(t^2-1)^2 P_n^2} dt\\
I_3 &= \int t(t^2-1)^4 Q_n^2 \left(P_n \dot{Q}_n - Q_n \dot{P}_n\right) dt
\end{aligned}$$

Let us now study this expression term by term. We begin with the third summand of (4.13) which is

$$2c\int \left(t^2-1\right)^3 Q_n^4\, dt.$$

It has already the form of Lemma 21. So as in the proof of Lemma 21, the monodromy commutator will be computed using

$$\operatorname*{Res}_{t=\infty}\,(t^2-1)^3(Q_n+\epsilon_n\alpha P_n)^4.$$

Now look at the term in b^2. It is not as complicated as we could think because of the following relation

$$P_n\dot{Q}_n-\dot{P}_nQ_n=(t^2-1)^{-2}\quad \forall n\in\mathbb{N}$$

which is linked to the Wronskian of Equation (4.10). Thanks to that, the term in b^2 can be written as

$$8b^2\int P_n^2Q_n(t^2-1)^2\int \frac{\int (t^2-1)^2Q_n^3+2\frac{\int P_nQ_n^2(t^2-1)^2dt}{(t^2-1)^2P_n^2}dt}{(t^2-1)^2P_n^2}dt\,dt$$

and then using integration by parts, this gives

$$16b^2\int Q_n^3(t^2-1)^2\int P_nQ_n^2(t^2-1)^2dt\,dt-8b^2\int P_nQ_n^2(t^2-1)^2dt\int Q_n^3(t^2-1)^2dt$$

Now by Lemma 22 we have for all even integers $n>1$:

$$\int P_nQ_n^2(t^2-1)^2dt,\ \int Q_n^3(t^2-1)^2dt\in\mathbb{C}(t)\left[\operatorname{arctanh}\left(\frac{1}{t}\right)\right].$$

So we are integrating a polynomial in arctanh with rational coefficients, and this corresponds to the hypotheses of Lemma 21. The second term does not provide any monodromy, so we only have to study the first term and thus the sequence

$$\operatorname*{Res}_{t=\infty}\,(Q_n+\epsilon_n\alpha P_n)^3(t^2-1)^2\int P_n(Q_n+\epsilon_n\alpha P_n)^2(t^2-1)^2.$$

Now we look at the term in a^2. It can be simplified to

$$\int 2a^2(t^2-1)^2P_nQ_nt\int \frac{\int \frac{(t^2-1)^2Q_n^3t}{P_n}+\frac{\int (t^2-1)^2Q_n^2t\,dt}{(t^2-1)^2P_n^2}dt}{t^2(t^2-1)^2}+\frac{\int\frac{\int (t^2-1)^2Q_n^2t\,dt}{t^2(t^2-1)^2}dt}{(t^2-1)^2P_n^2}dt$$

We now use again integrations by parts (recall that $P_2=4t$):

$$8a^2\int (t^2-1)^2Q_n^2Q_2\int (t^2-1)^2Q_n^2t\,dt-8a^2\int (t^2-1)^2Q_n^2t\int (t^2-1)^2Q_n^2Q_2dt.$$

To conclude we can again use Lemmas 21 and 22. We first prove that

$$\forall n\neq 1\quad \int P_2Q_n^2(t^2-1)^2dt,\ \int Q_n^2Q_2(t^2-1)^2dt\in\mathbb{C}(t)\left[\operatorname{arctanh}\left(\frac{1}{t}\right)\right].$$

The case $n=1$ corresponds to $\lambda=0$, for which we have always the coefficient $a=0$. Now we make a final integration by parts which gives

$$16a^2\int (t^2-1)^2Q_n^2Q_2\int (t^2-1)^2Q_n^2t\,dt\,dt-8a^2\int (t^2-1)^2Q_n^2t\,dt\int (t^2-1)^2Q_n^2Q_2dt.$$

Thanks to that, we get a constraint of the form given by Theorem 17 and the coefficients are given by (multiplying them by ϵ_n^{-2} for further simplifications)

$$f_1(n) = \langle \alpha^3 \rangle\, 2\epsilon_n^{-2} \operatorname*{Res}_{t=\infty} (t^2-1)^2 (Q_n + \epsilon_n \alpha P_n)^2 (Q_2 + \epsilon_2 \alpha P_2) \int (t^2-1)^2 (Q_n + \epsilon_n \alpha P_n)^2 P_2 dt, \tag{4.14}$$

$$f_2(n) = \langle \alpha^3 \rangle\, 2\epsilon_n^{-2} \operatorname*{Res}_{t=\infty} (t^2-1)^2 (Q_n + \epsilon_n \alpha P_n)^3 \int (t^2-1)^2 (Q_n + \epsilon_n \alpha P_n)^2 P_n dt, \tag{4.15}$$

$$f_3(n) = \langle \alpha^3 \rangle\, \frac{1}{6} \epsilon_n^{-2} \operatorname*{Res}_{t=\infty} (t^2-1)^3 (Q_n + \epsilon_n \alpha P_n)^4, \tag{4.16}$$

where $\langle \cdot \rangle$ denotes coefficient extraction. In fact, only the coefficient of α^3 appears in these residues. We need not to prove this fact, because we simply select the coefficient of α^3, ignoring the question whether the other coefficients are zero or not.

We now look at the case $n = 0$. All our previous calculations are also valid in this case except those involving Lemma 22 because we only have

$$\int P_0 Q_0^2 (t^2-1)^2 dt,\ \int Q_0^3 (t^2-1)^2 dt \in \mathbb{C}(t) \left[\operatorname{arctanh}\left(\frac{1}{t}\right), \ln(t^2-1) \right],$$

$$\int P_2 Q_0^2 (t^2-1)^2 dt,\ \int Q_0^2 Q_2 (t^2-1)^2 dt \in \mathbb{C}(t) \left[\operatorname{arctanh}\left(\frac{1}{t}\right) \right].$$

So, the coefficients in a^2, c are also

$$2 \operatorname*{Res}_{t=\infty} (t^2-1)^2 Q_0^2 Q_2 \int (t^2-1)^2 P_2 Q_0^2 \, dt, \qquad \frac{1}{6} \operatorname*{Res}_{t=\infty} \left((t^2-1)^3 Q_0^4 \right).$$

We find that these residues are both 0, and so the corresponding integral does not provide any additional monodromy. The case of the coefficient in b^2 is a little more difficult because the integral does not satisfy the conditions of Lemma 21. After an explicit computation, we arrive at the following integral

$$\int \frac{1}{t^2-1} \left(-t \operatorname{arctanh}\left(\frac{1}{t}\right) - \frac{1}{2} \ln\left(t^2-1\right) \right) dt =$$
$$\frac{1}{2} \ln(2) \ln(t-1) + \frac{1}{2} \operatorname{dilog}(t+1) + \frac{1}{8} \ln(t+1)^2 + \frac{1}{4} \ln(t+1) \ln(t-1) - \frac{1}{8} \ln(t-1)^2.$$

All the terms are in $\mathbb{C}[t, \operatorname{arctanh}\left(\frac{1}{t}\right), \ln\left(t^2-1\right)]$ except one, namely the dilogarithmic term

$$\operatorname{dilog}(t+1) = \int \frac{\ln(t+1)}{t} dt.$$

With the same idea as in Lemma 21, we see that this term has a noncommutative monodromy because of the following residue in 0

$$\operatorname{Res}_{t=0} \frac{\ln(t+1) + \alpha}{t} = \alpha$$

which depends explicitly on α. So, for $n = 0$, the integrability condition at order 3 is in fact just $b^2 = 0$.

□

4.6 Holonomicity and Asymptotics

In this section we are going to derive P-finite recurrences (i.e., linear recurrences with polynomial coefficients) for the sequences $f_1(n)$, $f_2(n)$, and $f_3(n)$ that appeared in section 4.5. The methods that we employ are based on Zeilberger's holonomic systems approach [75]. The recurrences presented below were computed with the method of creative telescoping, to which a brief introduction is given below (see [38] for more details).

Let S_n denote the forward shift operator in n, i.e., $S_n f(n) = f(n+1)$, and D_x the derivative w.r.t. x, i.e., $D_x f(x) = f'(x)$. The method works for the class of holonomic functions, which in short are (multivariate) functions that are solutions of maximally overdetermined systems of linear difference and differential equations with polynomial coefficients. The set of all equations which a given holonomic function satisfies forms a left ideal (we call it *annihilating ideal*) in some Ore algebra of the form

$$\mathbb{C}(m, n, \ldots, x, y, \ldots)\langle S_m, S_n, \ldots, D_x, D_y \ldots\rangle.$$

The nice fact about holonomic functions is that this class is closed under certain operations (addition, multiplication, certain substitutions, definite summation and integration) which can be executed algorithmically: given the defining systems of equations for two holonomic functions f and g, there are algorithms to compute a holonomic system for $f+g$, $f \cdot g$, etc.

For computing integrals (or residues), the method of creative telescoping makes use of the fundamental theorem of calculus. Consider a definite integral of the form $\int_a^b f\,dx$ where the integrand f depends also on some other (discrete and/or continuous) parameters. We need f to be holonomic, i.e., there is some left ideal I of annihilating operators in the corresponding Ore algebra $\mathbb{O}$. The idea is now to come up with an operator $A + D_x B \in I$ where $A, B \in \mathbb{O}$ and A does not depend on x and D_x (the concept of Gröbner bases [15] plays a crucial rôle in this step). Then after integration we get,

$$P \int_a^b f\,dx + \Big[Qf\Big]_a^b = 0,$$

in other words, we found a (possibly inhomogeneous) equation for the integral in question. The examples below will demonstrate this methodology clearly; we start with the simplest one, the sequence $f_3(n)$.

Lemma 23. *The sequence $f_3(n)$ given in* (4.16)*, satisfies the P-finite recurrence*

$$\begin{aligned}
&(4n+11)(4n+9)(n+1)^3(n+3)^2 f_3(n+2) - \\
&(2n+3)(16n^6+144n^5+515n^4+930n^3+888n^2+423n+81) f_3(n+1) + \\
&(4n+3)(4n+1)(n+2)^3 n^2 f_3(n) = 0.
\end{aligned}$$

subject to the initial conditions

$$f_3(1) = -\frac{8}{105}, \quad f_3(2) = -\frac{8}{385}.$$

Proof. It is an easy exercise to compute the first values of $f_3(n)$ explicitly with a computer algebra system. Thus we basically have to derive the recurrence. For this purpose, we compute an annihilating ideal I for $(t^2-1)^3(Q_n + \epsilon_n \alpha P_n)^4$ which is the expression in the residue (4.16). For this purpose we apply holonomic closure properties (note that $Q_n + \epsilon_n \alpha P_n$ satisfies the same equations as Q_n itself). The resulting Gröbner basis is too large to be printed here, namely a full page of equations approximately. It is represented in the Ore algebra $\mathbb{C}(n,t)\langle S_n, D_t\rangle$. In the next step we make use of a special algorithm [39] for computing a creative telescoping operator

$$A(n, S_n) + D_t B(n, t, S_n, D_t) \in I$$

(its existence is guaranteed by the theory of holonomy). Because we are dealing with a residue we can forget about the part B and find that A annihilates the residue. In order to obtain $f_3(n)$ we need to multiply the residue with $2\epsilon_n^{-2}$, which can be done again by closure properties. The resulting operator represents exactly the above recurrence. All these computations were done with the above mentioned package `HolonomicFunctions` [38, 40]. □

Lemma 24. *The sequence $f_1(n)$ given in (4.14) satisfies the P-finite recurrence*

$$\begin{aligned}&(4n+11)(4n+9)(n+4)^2(n+1)^3(4n^2+8n-9)f_1(n+2)-\\&(2n+3)(64n^8+768n^7+3580n^6+8028n^5+8113n^4+\\&\quad 834n^3-4863n^2-3276n-648)f_1(n+1)+\\&(4n+3)(4n+1)(n+2)^3(n-1)^2(4n^2+16n+3)f_1(n)=0\end{aligned}$$

subject to the initial conditions

$$f_1(2)=\frac{16}{1155},\quad f_1(3)=\frac{16}{2145}.$$

Proof. The proof is based on the same ideas as in Lemma 23, except that the expression of which we have to take the residue is more complicated. In particular, an indefinite integral occurs (recall that indefinite integration is not among the holonomic closure properties) and it is not clear a priori how to choose the integration constant such that the result is again holonomic. We start by computing an annihilating ideal I for

$$F(n,t)=(t^2-1)^2(Q_n+\epsilon_n\alpha P_n)^2.$$

Thus for all $A\in I$ the operator AD_t annihilates the indefinite integral $\int F(n,t)\,dt$. Additionally, from a creative telescoping operator $A+D_tB\in I$ we can derive more such annihilating operators. Let J denote the annihilating ideal for $B(F)$ which can be obtained by holonomic closure properties. Then for every $C\in J$, the operator CA annihilates the indefinite integral as well. Altogether we obtain a zero-dimensional annihilating ideal for $\int F(n,t)\,dt$, and continue as in Lemma 23. □

These recurrences in Lemmas 23 and 24 are irreducible (in the sense that the corresponding operator cannot be factorized), and so we are not able to find closed forms for f_1 and f_3. The recurrence for $f_2(2n)$ is given by a third-order recurrence with polynomial coefficients of degree larger than 50, which we do not state here explicitly. The initial conditions are

$$f_2(2)=\frac{16}{1155},\quad f_2(4)=\frac{184}{183141},\quad f_2(6)=\frac{38308}{181081875}.$$

This recurrence is reducible and possesses a hypergeometric solution

$$f_2(2)\frac{8\pi^2\Gamma(n+1)\Gamma(5/6+n)^2\Gamma(1/6+n)^2\Gamma(n)^3}{25\Gamma(n+2/3)^2\Gamma(3/2+n)^3\Gamma(1/2+n)\Gamma(4/3+n)^2}$$

but because $f_2(2)\neq 0$, the recurrence for $f_2(2n)$ cannot be reduced.

We are interested in a **practical** way to apply the third-order variational equation. To do this, these recurrences are not enough, since we need closed forms. As these closed forms do not exist, we will instead produce closed form expressions which approach f_1, f_2, and f_3 with a controlled relative error. In the following, we will denote the harmonic numbers by

$$H(n)=\sum_{i=1}^{n-1}\frac{1}{i}.$$

Definition 18. *Let us consider an operator $L \in \mathbb{C}\langle n, S_n\rangle$, in other words L represents a linear recurrence with polynomial coefficients. We will say that L is regular at infinity if for all solutions u (i.e., $Lu = 0$) there exist $\alpha \in \mathbb{Z}$, $\beta \in \mathbb{N}$, and $\gamma \in \mathbb{C}$ such that*

$$u(n) \sim \gamma n^\alpha H(n)^\beta \quad \text{for } n \to \infty.$$

Theorem 22. *Consider $L \in \mathbb{C}\langle n, S_n\rangle$ of order k and assume that it is regular at infinity. Then for all $p \in \mathbb{N}$ and for all u solution of $Lu = 0$, there exists a function $F \in \mathbb{C}(n)[H(n)]$ with degree in $H(n)$ less than $k-1$ such that*

$$u(n) = F(n) + O\left(\frac{H(n)^{k-1}}{n^p}\right).$$

This theorem is directly implied by the theorem of Birkoff given in [72], which gives a general form of an asymptotic expansion which is always possible. In our case, we will only use what we call the regular case, which in a Birkoff expansion corresponds to not having an exponential part.

Definition 19. *Consider a function $f : \mathbb{N} \longrightarrow \mathbb{R}$ and a function $F \in \mathbb{R}(n)[H(n)]$. We say that F is an approximation of f with relative error ϵ at rank n_0 if*

$$\left|\frac{f(n)}{F(n)} - 1\right| \le \epsilon \quad \forall n \ge n_0.$$

We consider p functions $f_1, \dots, f_p : \mathbb{N} \longrightarrow \mathbb{R}$ and approximations $F_1, \dots, F_p \in \mathbb{R}(n)[H(n)]$ with relative error ϵ at rank n_0. We define the error amplification factor A by

$$A = \min\left\{ \tilde{A} \in \mathbb{R}_+^* \text{ such that } \left|\frac{\sum_{i=1}^p f_i(n)}{\sum_{i=1}^p F_i(n)} - 1\right| \le \tilde{A}\epsilon \quad \forall n \ge n_0 \right\}.$$

Lemma 25. *We consider p functions $f_1, \dots, f_p : \mathbb{N} \longrightarrow \mathbb{R}$ and approximations $F_1, \dots, F_p \in \mathbb{R}(n)[H(n)]$ with relative error $\epsilon < 1$ at rank n_0 and A their amplification factor. Then*

$$A \le \max_{n \ge n_0} \frac{\sum_{i=1}^p |F_i(n)|}{\left|\sum_{i=1}^p F_i(n)\right|}.$$

Proof. The lemma is equivalent to prove that

$$\left|\frac{\sum_{i=1}^p f_i(n)}{\sum_{i=1}^p F_i(n)} - 1\right| \le \epsilon \max_{n \ge n_0} \frac{\sum_{i=1}^p |F_i(n)|}{\left|\sum_{i=1}^p F_i(n)\right|}$$

So one just needs to maximize the left hand side. We already know that $|f_i(n)/F_i(n) - 1| \le \epsilon$. So depending on the sign of $f_i(n)$ we replace $f_i(n)$ by $(1-\epsilon)F_i(n)$ or $(1+\epsilon)F_i(n)$. We then expand

$$\left|\frac{\sum_{i=1}^p f_i(n)}{\sum_{i=1}^p F_i(n)} - 1\right| \le \left|\frac{\epsilon \sum_{i=1}^p \operatorname{sign}(f_i(n)) F_i(n)}{\sum_{i=1}^p F_i(n)}\right| \le \left|\frac{\epsilon \sum_{i=1}^p |F_i(n)|}{\sum_{i=1}^p F_i(n)}\right| \le \epsilon \max_{n \ge n_0} \frac{\sum_{i=1}^p |F_i(n)|}{\left|\sum_{i=1}^p F_i(n)\right|}$$

using the fact that $f_i(n)$ and $F_i(n)$ have always the same sign for $n \ge n_0$ (because $\epsilon < 1$). $\square$

In practice, we first check that the sign of the functions $F_i(n)$ and their sum does not change for $n \geq n_0$ and then we prove a majoration of the resulting expression in $\mathbb{R}(n, H(n))$. So all comes down to prove that some polynomial in $\mathbb{R}[n, H(n)]$ does not vanish for $n \geq n_0$. This can be done by first making an encadrement of the function $H(n)$ and then prove that the corresponding bivariate polynomial does not vanish on a particular algebraic subset. Such a problem can be algorithmically decided.

Theorem 23. *Consider the recurrence equation*

$$u(n+1) = A(n)u(n) \quad \forall n \in \mathbb{N},\ A(n) \in M_p(\mathbb{C}) \tag{4.17}$$

Consider $\|\cdot\|$ *a matricial norm and* $R(n)$ *the resolvant matrix of equation* (4.17). *Assume that*

$$M(\infty) = \sum_{j=0}^{\infty} \|A(j) - I_p\| \ < 1.$$

Then

$$\|R(n) - I_p\| \leq \frac{M(\infty)}{1 - M(\infty)} \quad \forall n \in \mathbb{N}.$$

Proof. We write

$$R(n) = \prod_{i=0}^{n-1} A(i) = \prod_{i=0}^{n-1} ((A(i) - I_p) + I_p).$$

Let us pose

$$M(n) = \sum_{j=0}^{n-1} \|A(j) - I_p\|.$$

We want to prove a majoration of the type

$$\|R(n) - I_p\| \leq CM(n) \tag{4.18}$$

with a suitable constant $C > 0$. For $n = 1$, this is true with $C = 1$. Let us prove equation (4.18) by recurrence:

$$R(j) = \prod_{i=0}^{j-1} ((A(i) - I_p) + I_p) = (A(j-1) - I_p) \prod_{i=0}^{j-2} A(i) + \prod_{i=0}^{j-2} A(i),$$

$$R(j) - R(j-1) = (A(j-1) - I_p) \prod_{i=0}^{j-2} A(i) = (A(j-1) - I_p)R(j-1).$$

Then we sum these equations for $1 \leq j \leq n$ which produces

$$\begin{aligned}
\|R(n) - I_p\| &= \left\| \sum_{j=0}^{n-1} (A(j) - I_p)(R(j) - I_p) + (A(j) - I_p) \right\| \\
&\leq \sum_{j=0}^{n-1} \|A(j) - I_p\| \, \|R(j) - I_p\| + \|A(j) - I_p\| \\
&\leq M(n) + \sum_{j=0}^{n-1} \|A(j) - I_p\| \, CM(j) \\
&= M(n) + CM(n)^2 \\
&\leq (1 + CM(\infty))M(n)
\end{aligned}$$

using the fact that $M(n)$ is a growing sequence. So the recurrence property is proved if $C \leq 1 + CM(\infty)$ which is equivalent to $C \geq (1 - M(\infty))^{-1} \geq 1$. So this proves that

$$\|R(n) - I_p\| \leq \frac{M(n)}{1 - M(\infty)} \leq \frac{M(\infty)}{1 - M(\infty)}$$

which proves the theorem. □

The main application of this theorem is to compute a sequence with controlled error. Let us take an operator $L \in \mathbb{R}\langle n, S_n \rangle$ regular at infinity. We can then compute an asymptotic expansion of the resolvant matrix of L, and an error matrix which will satisfy an equation like (4.17). Then for an $n_0 \in \mathbb{N}$, we can apply Theorem 23 for the shifted sequence $u(n + n_0)$, and the majoration $M(\infty)$ will become very small for n_0 big enough, giving us that the error is always lower than some explicit bound. This has very important consequences for the application of the higher variational method. In particular, it becomes possible to rigorously prove that a sequence of potentials with the unbounded eigenvalue property does not satisfy integrability criteria for λ large enough, and thus coming back to a bounded eigenvalue problem.

4.7 Application at Order 2

We now apply the second order criterion to our example. We begin with the case E_4. Before we state the corresponding theorem, we need a preparatory lemma concerning the solutions of a certain Diophantine equation.

Lemma 26. *The set of solutions* $(k_1, k_2) \in \mathbb{N}^2$ *of the Diophantine equation*

$$R(k_1, k_2) = k_2^2 k_1^2 + k_2 k_1^2 - 75k_1^2 - 75k_1 + k_2 k_1 - 27k_2 + k_2^2 k_1 - 27k_2^2 = 0$$

is given by $\{(0, 0), (6, 14)\}$.

Proof. We begin by proving that for $k_2 \geq 50$, the condition $R = 0$ implies $4 < k_1 < 5$, and similarly, for $k_1 \geq 50$, we have $8 < k_2 < 9$. These statements can be written as logical expressions involving polynomial inequalities

$$\forall k_1 \forall k_2 : (k_1 \geq 0 \wedge k_2 \geq 50 \wedge R(k_1, k_2) = 0) \implies 4 < k_1 < 5, \qquad (4.19)$$
$$\forall k_1 \forall k_2 : (k_1 \geq 50 \wedge k_2 \geq 0 \wedge R(k_1, k_2) = 0) \implies 8 < k_2 < 9. \qquad (4.20)$$

Such formulas can be proven routinely with quantifier elimination techniques like cylindrical algebraic decomposition [18]. Indeed, applying the Mathematica command **CylindricalDecomposition** to the above formulae reveals that they are true. Therefore, there are no integer solutions for $k_1 \geq 50$ or $k_2 \geq 50$ and an exhaustive search delivers exactly the solutions claimed above (Figure 4.7.

However, if we want to prove (4.19) and (4.20) "by hand" (let's consider the first one for the moment), we have to look at the largest real root of the polynomial

$$\mathrm{res}_{k_1}\left(R(k_1, k_2), \frac{\partial R(k_1, k_2)}{\partial k_1}\right) R(4, k_2)\, R(5, k_2).$$

We find that this root is smaller than 50 (using real root isolation) and that the limit

$$\lim_{k_2 \to \infty} \kappa(k_2) = -\frac{1}{2} + \frac{1}{2}\sqrt{109}$$

is between 4 and 5, where $\kappa(k_2)$ denotes the positive solution of $R(k_1, k_2) = 0$ regarded as an equation in k_1. The implication (4.19) follows, and (4.20) can be proven analogously. □

Theorem 24. *We consider the potential E_4 given in Theorem 20. If the variational equation near all Darboux points is integrable at order 2, then the corresponding eigenvalues are integers of the form $\lambda = (2l-1)(l+1),\ \ l \in \mathbb{N}$.*

Proof. We use the notation $U = rE_4$ from Theorem 20. The condition $U'(\theta) = 0$ yields the two Darboux points

$$c_1 : e^{i\theta} = 1, \qquad c_2 : e^{i\theta} = \frac{s + 6\lambda_1}{s - 6\lambda_2}. \tag{4.21}$$

There are singular cases of the second equation, namely for $s + 6\lambda_1 = 0$ or $s - 6\lambda_2 = 0$. After solving and replacing, we find that these cases correspond exactly to $k_1 = 0$ and $k_1 = 3$, which were excluded from E_4.

We now compute the third derivative of V, evaluated at the two Darboux points c_1 and c_2 given by expression (4.21):

$$\frac{\partial^3 V}{\partial q_2^3}(c_1) = \frac{i\lambda_1(s + 15\lambda_1 + 9\lambda_2)}{\lambda_1 + \lambda_2} \qquad \frac{\partial^3 V}{\partial q_2^3}(c_2) = -\frac{i\lambda_2(s - 15\lambda_2 - 9\lambda_1)}{3(\lambda_1 + \lambda_2)}.$$

In the case (k_1, k_2) both odd, both derivatives should vanish. We solve the system and we find $4i(k_2 + 1)k_2 = 0$. This is impossible for odd values. In the case k_1 odd k_2 even, the first one should vanish, and in the case k_1 even k_2 odd the second one should vanish. We get the equations

$$\begin{aligned} &\frac{k_1^2(k_1+1)^2(k_2^2k_1^2 + k_2^2k_1 - 27k_2^2 - 27k_2 - 75k_1 + k_2k_1 - 75k_1^2 + k_2k_1^2)}{12(k_2^2 + k_2 + k_1 + k_2^2)} \\ &\frac{k_2^2(k_2+1)^2(k_1^2k_2^2 + k_1^2k_2 - 27k_1^2 - 27k_1 - 75k_2 + k_1k_2 - 75k_2^2 + k_1k_2^2)}{12(k_1^2 + k_1 + k_2 + k_2^2)} \end{aligned} \tag{4.22}$$

These two conditions are symmetric. The first terms can never vanish because we have k_1 odd for the first one and k_2 odd for the second one. To conclude, we need to look at the last term, which corresponds to a Diophantine equation, and to prove that this equation does not have a solution with k_1 odd and k_2 even.

With Lemma 26, we have no solutions from the second term where k_1 and k_2 have different parity. We conclude that all the possibilities left are for k_1, k_2 even. □

It is well known that Diophantine equations in general cannot be solved (Matiyasevich's theorem). This means that Lemma 26 is a lucky case, although not trivial to prove. We therefore should remark that the study of this equation is not absolutely mandatory. We could simply skip it, **assume** that it is satisfied and continue further to the third-order condition. This condition would add two additional equations in k_1 and k_2 and thus would allow to solve the problem in all generality.

Here we are in a special case. A Diophantine equation $R(k_1, k_2) = 0$ can be solved only using real algebraic geometry in one of the following cases:

1. The set $R^{-1}(0) \cap \mathbb{R}^{+2}$ is compact. In this case we only have a finite number of points to test.

2. The set $R^{-1}(0) \cap \mathbb{R}^{+2}$ is not compact but all infinite branches are asymptotes and the corresponding asymptotic straight lines have a rational slope. In this case, either R is homogeneous and has an infinite number of solutions, or the integer solutions can be bounded: when approaching infinity, the infinite branch of $R^{-1}(0)$ comes closer to the asymptotic line without touching it; for rational slope, there is then a nonzero infimum for the distance between the asymptotic straight line and integer points).

The first case can be considered to be part of the second one with no asymptotes at all. In Lemma 26, we encounter the second case.

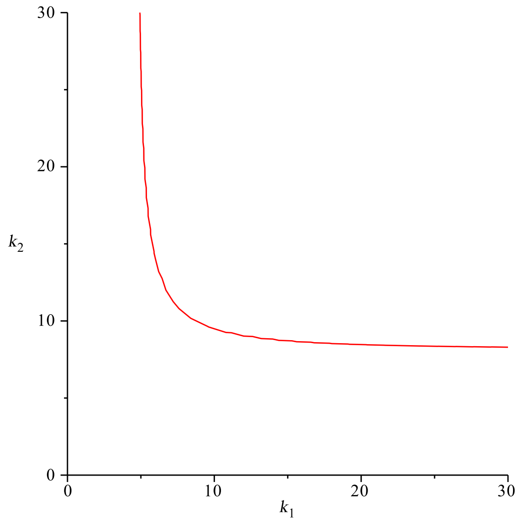


Figure 4.1: Graph of $R^{-1}(0)$. The graph $R^{-1}(0) \cap \mathbb{R}^{+2}$ is not compact but the infinite branches are asymptotes with rational or vertical slopes; here the asymptotes are $k_1 + \frac{1}{2} - \frac{1}{2}\sqrt{109} = 0$ and $k_2 + \frac{1}{2} - \frac{1}{2}\sqrt{301} = 0$.

Remark 12. *The potential corresponding to $k_1, k_2 = (6, 14)$ is the following (with the good choice of valuation for the square root):*

$$V(r,\theta) = \frac{1}{r}\left(-20 + \frac{105}{2}e^{i\theta} - 42e^{2i\theta} + \frac{21}{2}e^{3i\theta}\right).$$

This potential has two Darboux points, it is integrable at order 2 near these two Darboux points and we have also that the third derivative near one of the Darboux points is zero (which is not needed for integrability at order 2 but gives interesting properties in practice at order 3).

Theorem 25. *Among the potentials in the families E_1, E_2, E_3, if a potential V is meromorphically integrable, then it is of the form (after multiplying by some constant factor):*

$$V = \frac{1}{r}\left(-\frac{1}{3}k(2k+1)e^{3i\theta} + \frac{1}{2}k(2k+1)e^{2i\theta} - \frac{1}{6}(2k^2+k-6)\right),$$
$$V = \frac{1}{r}\left(-\frac{1}{2}k(2k+1)e^{2i\theta} + k(2k+1)e^{i\theta} - \frac{1}{2}(2k^2+k-2)\right),$$

for $k \in \mathbb{N}$.

Proof. The potentials E_2 and E_3 possess only one Darboux point. The corresponding potentials are

$$E_2: \; V = r^{-1}\left(-\frac{1}{6}k(k+1)e^{3i\theta} + \frac{1}{4}k(k+1)e^{2i\theta} - \frac{1}{12}k^2 - \frac{1}{12}k + 1\right),$$
$$E_3: \; V = r^{-1}\left(-\frac{1}{4}k(k+1)e^{2i\theta} + \frac{1}{2}k(k+1)e^{i\theta} - \frac{1}{4}k^2 - \frac{1}{4}k + 1\right).$$

We know that if k is odd, we have an additional integrability condition at order 2. We find that

$$\frac{\partial^3 V}{\partial q_2^3}(c) = \frac{5}{2}ik(k+1) \text{ for } E_2, \qquad \frac{\partial^3 V}{\partial q_2^3}(c) = \frac{3}{2}ik(k+1) \text{ for } E_3.$$

These terms should vanish. This is never fulfilled for odd k. The sequence of potentials given by Theorem 25 corresponds exactly to the cases of even k (for which there is no condition for integrability at order 2). At last, we have the potential E_1. The corresponding eigenvalue is always -1, so it is always integrable at order 2. At order 3, we know that the integrability condition is $U^{(3)}(0) = 0$. We get

$$U^{(3)}(0) = -2ib$$

So the only possibility is $b = 0$ and this corresponds to the potential $V = r^{-1}$. This potential is integrable and already belongs to the family described by Theorem 25.

□

4.8 Application at Order 3

We will now prove Theorem 18, building an algorithm to prove it.

Proof. The scheme of the proof is the following

- First we prove that the recurrences for f_1, f_2, f_3 are regular at infinity.
- We then produce a series expansion $\tilde{R}_i(n)$ at infinity at an order high enough of the resolvant matrix $R_i(n)$ associated to these recurrences.
- We then write $R_i(n) = \tilde{R}_i(n)\tilde{R}_i(n_0)^{-1}R_i(n_0)E_i(n)$ for a large enough $n_0 \in \mathbb{N}$ and build a recurrence of the form (4.17) whose resolvant matrix is $E_i(n)$ (after change of basis), which will be denoted by $E_i(n+1) = A_i(n)E_i(n)$. We have moreover that $E_i(n_0)$ is the identity matrix.
- As $\tilde{R}_i(n)$ is a good approximation of $R_i(n)$ when $n \longrightarrow \infty$, the matrix $A_i(n)$ will tend to the identity matrix when $n \longrightarrow \infty$. Using Theorem 23 with a shift in the indices, we will have that

$$\|E_i(n) - I\| \leq \frac{\sum\limits_{j=n_0}^{\infty} \|A_i(j) - I\|}{1 - \sum\limits_{j=n_0}^{\infty} \|A_i(j) - I\|} \quad \forall n \geq n_0$$

- If we have chosen an expansion order and n_0 large enough, this sum will be finite and small, and thus will give us an approximation of $R_i(n)$ by $\tilde{R}_i(n)$ with relative error control. The expressions in Theorem 18 follow.

For $f_3(2n)$, we find the following asymptotic expansion (a high order makes up the computation easier for error control)

$$c_1\left(\frac{1}{n^4} - \frac{1}{n^5} + \frac{25}{32n^6} - \frac{35}{64n^7} + \frac{183}{512n^8}\right) +$$
$$c_2\left(\left(\frac{3}{16n^4} - \frac{3}{16n^5} + \frac{75}{512n^6} - \frac{105}{1024n^7} + \frac{549}{8192n^8}\right) H(n) + \right.$$
$$\left.\frac{1}{n^2} - \frac{1}{2n^3} + \frac{19951}{46848n^4} - \frac{7507}{46848n^5} + \frac{96541}{1499136n^6} - \frac{58151}{2998272n^7}\right)$$

This proves by the way that the recurrence for $f_3(2n)$ is regular. We do the same for $f_1(2n)$ and $f_2(2n)$ and we find that they are regular too. We then find a majoration of the norm of the error matrix $A_3(n)$

$$\|A_3(n)\|_\infty \leq \frac{9975}{256n^6} + \frac{29925}{4096}\frac{H(n)}{n^6} + \frac{9975}{256n^8} + \frac{29925}{4096}\frac{H(n)}{n^8}$$

We choose now $n_0 = 100$. We majorate the sum of this majoration beginning at $n = 100$. We find a majoration of this sum by

$$\sum_{n=100}^{\infty} \|A_3(n)\|_\infty \leq 4.84522 \times 10^{-9}$$

$$\|E_3(n)\| \leq \frac{4.84522 \times 10^{-9}}{1 - 4.84522 \times 10^{-9}} \qquad \forall n \geq n_0$$

(an explicit rational number). We then compute the recurrence up to $n = 100$, and then produce an encadrement (with error less than 10^{-10}) of the result with rational numbers. Although it is not mandatory in theory, in practice recurrences tend to produce very large rational numbers, whose size grows linearly with n, and thus are impractical to manipulate. This gives us the coefficients c_1, c_2 with a good error control:

$$c_1 = -\frac{883919839}{274877906944}, \qquad c_2 = -\frac{1740684681}{8589934592}.$$

We then compute the error amplification of the sum, and find that it is less than 33/32. As the resulting expression is too complicated to manipulate for applications, we only keep the terms up to order 3 and prove that this new approximation has a relative error less than 10^{-5}. The expressions for f_1 and f_2 are found with a similar way, with the exception that at the end, to produce a sufficiently simple and accurate formula, it is not sufficient to keep the terms up to order 2 (after there is a $H(n)$ that we want to avoid), so we need to add a term of order 3 (without $H(n)$) with a well chosen coefficient such that the error stays below 10^{-5} (else the result is only accurate to 10^{-3}). □

Theorem 26. *The third order integrability conditions for the families*

$$V = \frac{1}{r}\left(-\frac{1}{3}k(2k+1)e^{3i\theta} + \frac{1}{2}k(2k+1)e^{2i\theta} - \frac{1}{6}(2k^2+k-6)\right)$$
$$V = \frac{1}{r}\left(-\frac{1}{2}k(2k+1)e^{2i\theta} + k(2k+1)e^{i\theta} - \frac{1}{2}(2k^2+k-2)\right)$$

where $k \in \mathbb{N}^*$, are

$$9(k+1)^2(2k-1)^2 f_1(2k) = 25k^2(2k+1)^2 f_2(2k) + (66k^2+33k-9)f_3(2k),$$
$$9(k+1)^2(2k-1)^2 f_1(2k) = 9k^2(2k+1)^2 f_2(2k) + (42k^2+21k-9)f_3(2k),$$

respectively. They are never satisfied.

Proof. We replace $f_1(2k), f_2(2k), f_3(2k)$ by their approximations, and then compute the error amplification. It is less than 33/32, and the resulting expression does not vanish for $k \geq 100$. For $k < 100$, we make exhaustive testing and we do not find any solutions. For the second equation, we do not find any solution either. □

Theorem 27. *We consider the family of potentials E_4*

$$E_4: \quad V = r^{-1}\left(\frac{(s-6\lambda_2)\lambda_2}{18(\lambda_1+\lambda_2)}e^{3i\theta} - \frac{(3\lambda_1+s-3\lambda_2)\lambda_2}{6(\lambda_1+\lambda_2)}e^{2i\theta} + \frac{(6\lambda_1+s)\lambda_2}{6(\lambda_1+\lambda_2)}e^{i\theta} + \frac{-9\lambda_1\lambda_2 - \lambda_2 s + 18\lambda_1 + 18\lambda_2 - 3\lambda_2^2}{18(\lambda_1+\lambda_2)}\right)$$

with

$$s^2 = 6\lambda_1^2\lambda_2 + 6\lambda_1\lambda_2^2 - 36\lambda_1\lambda_2 \qquad \lambda_1 = \frac{1}{2}(k_1-1)(k_1+2)+1$$

$$\lambda_2 = \frac{1}{2}(k_2-1)(k_2+2)+1 \quad k_1,k_2 \in \mathbb{N}^* \;\; k_1 \neq 3$$

The third order integrability condition for E_4 is of the form

$$Q_{k_1,k_2}(f_1(k_1),f_2(k_1),f_3(k_1)) = 0 \qquad Q_{k_2,k_1}(f_1(k_2),f_2(k_2),f_3(k_2)) = 0$$

where Q is a quadratic form depending polynomially on k_1 and k_2.

Proof. We use Theorem 17 and compute the derivatives of the potentials in the family E_4. These derivatives depend rationally on k_1, k_2, and s. As there are two Darboux points, we get two conditions $(C_1),(C_2)$ linearly dependent on $f_1(k_1),f_2(k_1),f_3(k_1)$ or $f_1(k_2),f_2(k_2),f_3(k_2)$ respectively for each Darboux point. To remove the quadratic extension s, we make the product $(C_1)\times \text{subs}(s=-s,(C_1))$ and $(C_2)\times \text{subs}(s=-s,(C_2))$. The fact that in the potentials of E_4, the two parameters λ_1 and λ_2 play a symmetric rôle produces the two conditions $Q_{k_1,k_2}=0$ and $Q_{k_2,k_1}=0$. □

Remark 13. *The conditions Q_{k_1,k_2},Q_{k_2,k_1} are not equivalent to the conditions $(C_1),(C_2)$. We can solve (C_1) in the quadratic extension and get for example that s should be rational because f_1,f_2,f_3 are always rational (this can be proven even without the P-finite recurrences since they correspond to a particular term in the series expansion of rational expressions in $t,P_n(t),Q_n(t)$). We get that*

$$\sqrt{3k_1k_2(k_2+1)(k_1+1)(k_1+k_1^2+k_2+k_2^2-12)} \;\; \in \mathbb{N} \tag{4.23}$$

if some generic condition depending on the $f_i(k_2),f_i(k_1)$ is satisfied. It corresponds to a Diophantine equation but it does not possess the nice properties we used to solve Lemma 26. We know moreover that (k_1,k_2) should be even. A direct search produces the picture given in Figure 4.2.

Theorem 28. *The third order integrability condition for E_4 is never satisfied except for $(k_1,k_2)=(2,2)$.*

Proof. Recall that the parameters (k_1,k_2) need to be both even for a potential E_4 to be integrable at order 2 near all Darboux points. We begin by solving $Q_{k_2,k_1}(f_1(k_2),f_2(k_2),f_3(k_2))=0$ in k_1. This is a polynomial of degree 4 in k_1 and as a polynomial, its Galois group is D_4. This allows us to write the solution in a relatively simple form

$$\begin{gathered} k_1 = -\frac{1}{2}+\sqrt{F_1(k_2)+wF_2(k_2)} \quad \text{with} \\ w^2 = 9(k_2+2)^2(k_2-1)^2 f_1(k_2)f_2(k_2) - 6(k_2+3)(k_2-2)f_2(k_2)f_3(k_2) + 36 f_3(k_2)^2 \end{gathered} \tag{4.24}$$

where $F_1,F_2 \in \mathbb{Q}(f_1,f_2,f_3,k_2)$. Moreover, k_1,k_2 are even integers. Let us prove that in fact, for even $k_2 \geq 200$, the expression

$$-\frac{1}{2}+\sqrt{F_1(k_2)+wF_2(k_2)}$$

is always complex for all possible valuations of the square roots. To have real values, we need that $F_1(k_2)+wF_2(k_2)$ be positive for at least one valuation of the square root. Let us begin by proving that w never vanishes. The function w^2 is a polynomial in $\mathbb{Q}[f_1,f_2,f_3,k_2]$. Thanks to Theorem 18, we can express f_1,f_2,f_3 in k_2 with controlled relative error. We check that the amplication of the error is small after summation of all terms (here it is less than $1+10^{-3}$) and that the approximated expressions never vanish. Now we need to prove that

$$F_1(k_2)+wF_2(k_2)<0 \text{ and } F_1(k_2)-wF_2(k_2)<0.$$

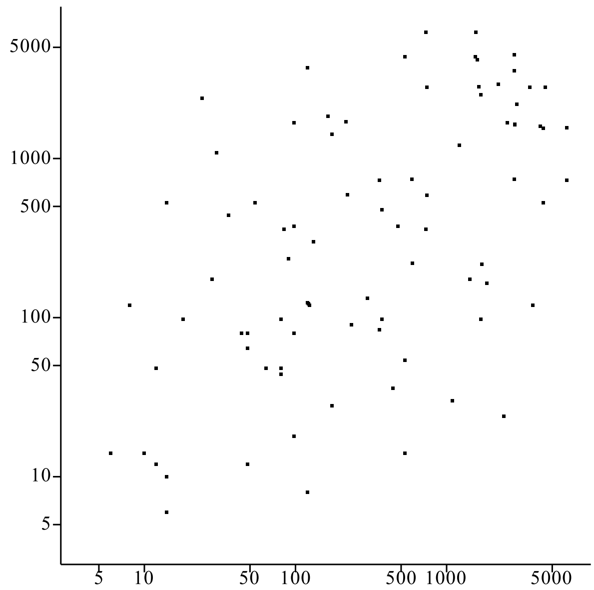


Figure 4.2: Each dot corresponds to a possible even (k_1, k_2). The solutions seems to be unbounded, and the set is probably Zarisky dense. Thus, no practical algebraic information can be extracted from this constraint. There are infinitely many solutions (because the diagonal part reduces to a Pell equation) with simultaneously arbitrarily high k_1, k_2. So here Theorem 17 is useless without Theorem 18.

We first prove that $F_2(k_2)$ and $F_1(k_2)$ (which are in $\mathbb{Q}[f_1, f_2, f_3, k_2]$ of degree $3, 4$ in f_i respectively) are always negative. Then we just have to prove that

$$\frac{F_1(k_2)}{wF_2(k_2)} > 1 \iff \frac{F_1(k_2)^2}{w^2F_2(k_2)^2} > 1 \iff F_1(k_2)^2 - w^2F_2(k_2)^2 > 0.$$

The last expression is in $\mathbb{Q}[f_1, f_2, f_3, k_2]$ (of degree 8 in f_i), so we can prove this statement. Again we compute the error amplification of the sum and it stays below $1 + 10^{-3}$, and the error is then still less than 10^{-4}. Eventually, we prove that this approximated expression never vanishes and is always positive. For the remaining cases, we use exhaustive testing and we find only one solution $(k_1, k_2) = (2, 2)$. $\square$

The case $(k_1, k_2) = (2, 2)$ corresponds to the third case of Theorem 19. It is really integrable with a quadratic in momenta additional first integral which is given in [31] page 107 case (8).

4.9 Remaining Cases and Conclusion

The remaining cases are the ones which do not posess a non-degenerate Darboux point.

Theorem 29. *Consider the set of potentials V given by* (4.6) *and assume that V does not possess a non-degenerate Darboux point c. If V is meromorphically integrable, then V belongs to one of the families*

$$V = \frac{1}{r}\left(a + be^{i\theta}\right), \qquad V = \frac{1}{r}\left(a + be^{2i\theta}\right),$$

$$V = \frac{1}{r}\left(a + be^{3i\theta}\right), \qquad V = \frac{1}{r}\left(a + be^{i\theta}\right)^3,$$

with $a \in \mathbb{C}$, $b \in \mathbb{C}^*$.

Proof. First let us suppose that V does not possess any Darboux point c. This means that the function

$$U(\theta) = a + be^{i\theta} + ce^{2i\theta} + de^{3i\theta}$$

does not possess any critical point. The only possibility is that $U(\theta) = F(e^{i\theta})$ with $F(z) = a + bz^n$, $b \neq 0$. This corresponds to the three first cases of Theorem 29. Now suppose there exists one Darboux point c but degenerate. After rotation, we can suppose that the Darboux point corresponds to $\theta = 0$. We have moreover the integrability constraint that $U''(0) = 0$. This gives the potential

$$V = \frac{a}{r}\left(e^{i\theta} - 1\right)^3.$$

After rotation, this corresponds to the fourth case of Theorem 29. □

Remark that in contrary to previous sections, we use the hypothesis that the first integrals are meromorphic on $\mathbb{C}^2 \times \mathcal{S}$, including $r = 0$. If we consider first integrals only meromorphic on $\mathbb{C}^2 \times \mathcal{S}^*$, the integrability constraint $U''(0) = 0$ does not hold anymore and two additional cases appear in Theorem 19

$$V = \frac{1}{r}\left(a + be^{i\theta}\right)^2, \qquad V = \frac{1}{r}\left(a + be^{i\theta}\right)^2\left(a - 2be^{i\theta}\right)$$

The family $V = \frac{1}{r}\left(a + be^{i\theta}\right)$ is integrable as given in [31], but with an additional first integral only meromorphic on $\mathbb{C}^2 \times \mathcal{S}^*$. For the other ones, the integrability status is still unknown. Let us remark now on the open cases. After rotation and dilatation, these cases correspond in fact to a finite number of potentials which are the following (the last two cases being open only for meromorphic first integrals on $\mathbb{C}^2 \times \mathcal{S}^*$):

$$\begin{aligned} &V = r^{-1}e^{2i\theta}, \qquad V = r^{-1}\left(e^{2i\theta} - 1\right), \, V = r^{-1}\left(e^{3i\theta} - 1\right), \\ &V = r^{-1}\left(e^{i\theta} - 1\right)^3, \, V = r^{-1}\left(e^{i\theta} - 1\right)^2, \, V = r^{-1}\left(e^{i\theta} - 1\right)^2\left(2e^{i\theta} + 1\right). \end{aligned} \tag{4.25}$$

We cannot study these potentials because we do not have a particular solution to study, or a sufficiently non-degenerate one (studying degenerate Darboux points with higher variational method is in fact useless and does not give any additional integrability condition). This is of course the main weakness of the Morales-Ramis theory. This is not due to the difficulty of applying the Morales-Ramis theory as we treat it in this chapter, but much more a fundamental limitation that seems hard to overcome. One approach could consist in looking for special algebraic orbits of these systems using a direct search (following Hietarinta [31]). This is not successful for all these potentials.

To conclude, let us remark that our holonomic approach to higher variational methods is very general, and in no way limited to this example. This could work at least for all problems about integrability of homogeneous potentials, as it allows to compute various higher integrability conditions of any fixed order. This is linked to the fact that the first order variational equation of a natural Hamiltonian system often corresponds to a spectral problem of a second order differential operator, which generates P-finite sequences of functions, which in turn appear in the study of higher variational equations. We could also wonder if these arbitrary high eigenvalues are really possible, and if this work is only conceptual and in practice useless. Indeed, very high eigenvalues should correspond to very high degree first integrals, and counting the number of conditions and number of free parameters for the existence of such high degree first integrals strongly suggests they do not exist. But this intuition is wrong, as Andrzej J. Maciejewski, Maria Przybylska found quite recently such an example in dimension 3. This is probably linked to the fact that most of integrable cases come from ultra-degenerate cases, as in our analysis: the generic case E_4 contains only one possibility, and when we look at the third order integrability condition, it seems really to be a miracle that this condition could ever be satisfied. On the contrary, the cases without Darboux points contain lots of integrable potentials.

Chapter 5

The case of non zero angular momentum

5.1 Introduction

In this chapter, our aim is to study dynamical systems of the form

$$\dot{q}_i = p_i \qquad \dot{p}_i = \frac{\partial}{\partial q_i} V \qquad i = 1 \dots n \tag{5.1}$$

where V is a homogeneous function in $q = (q_1, \dots, q_n)$ meromorphic for $q \in \mathbb{C}^n \setminus \{0\}$, and in particular the case of homogeneity degree -1 and its applications to celestial mechanics. In the following, we will call such a function a meromorphic homogeneous potential. This dynamical system (5.1) is also a Hamiltonian system which is given by

$$H(p,q) = \sum_{i=1}^{n} \frac{p_i^2}{2} - V(q)$$

One of the most important property in dynamical systems is integrability.

Definition 20. *Let V be a meromorphic homogeneous potential. We say that the dynamical system associated to the potential V is meromorphically integrable if there exists n functions $(p,q) \longrightarrow (I_1(p,q), \dots, I_n(p,q))$ meromorphic for $(p,q) \in \mathbb{C}^{2n}$, $q \neq 0$ such that*

- *The functions I_i are constant along orbits, meaning that $\dot{I}_i = 0$, $i = 1 \dots n$ (the functions I_i are then called first integrals)*
- *The functions I_i are in involution, meaning that for all $i, j = 1 \dots n$, we have the Poisson bracket $\{I_i, I_j\} = 0$*
- *The functions I_i are independent almost everywhere, meaning that the Jacobian matrix of the application $(p,q) \longrightarrow (I_1(p,q), \dots, I_n(p,q))$ has maximal rank almost everywhere*

Non-integrability of homogeneous potentials has been studied using mainly Morales Ramis theorem [50] and Ziglin theory [76]. These methods require a particular algebraic orbit of the corresponding potential. With homogeneous potentials, generically there exist straight line orbits, corresponding to the Darboux points of the potentials.

Definition 21. *(See for example [62, 43] on these equations) Let V be a meromorphic homogeneous potential. We say that $c \in \mathbb{C}^n \setminus \{0\}$ is a Darboux point if there exists $\alpha \in \mathbb{C}$ such that*

$$\frac{\partial}{\partial q_i} V(c) = \alpha c_i \quad \forall i = 1 \dots n$$

We call α the multiplier, and we say that c is non degenerate if $\alpha \neq 0$. A Darboux point c is also called a central configuration in the case of the n body problem.

To these Darboux points we can associate homothetic orbits (or more generally straight line orbits [32]), which are explicit algebraic solutions of the differential equation (5.1) (note that the orbit as a curve in $\mathbb{C}^{2n}$ is an algebraic curve, but not necessarily algebraic with respect to time). Using such orbits, Morales Ramis method provides a mean to prove some facts about non integrability, particularly in the case of homogeneous potentials.

Theorem 30. *([50] Theorem 4.1.) Let H be a Hamiltonian holomorphic on a complex symplectic manifold M of dimension $2n$, and $\Gamma \subset M$ a non-stationary orbit of H. If there are n meromorphic first integrals of H that are in involution and independent over a neighbourhood of Γ, then the identity component of Galois group of the variational equation near Γ is abelian.*

Theorem 31. *([53] Theorem 3.) Let V be a meromorphic homogeneous potential of degree -1 and c a Darboux point with multiplier -1. If V is meromorphically integrable, then*

$$Sp\left(\nabla^2 V(c)\right) \subset \left\{ \tfrac{1}{2}(k-1)(k+2),\ k \in \mathbb{N} \right\}.$$

Table 5.1: Integrability table for homogeneous potential of degree -1 invariant by rotation.

(C,H)	λ
$C=0$	$\lambda \in \left\{\frac{1}{2}(k-1)(k+2),\ k\in\mathbb{N}\right\}$
$C^2H=-1/2$	$\lambda \in \left\{-k^2,\ k\in\mathbb{N}\right\}$
$H=0$	$\lambda \in \left\{\frac{1}{2}(k-1)(k+2),\ k\in\mathbb{N}\right\}$
$C^2H \notin \{0,-1/2\}$	$\lambda \in \{0,-1\}$
$(C,H)=(0,0)$	$\lambda \in \mathbb{C}$

Early work on this subject has been done in [74, 73], similar statements were made and applied in [51, 52], generalizations were made for higher variational equations in [54] and for non Hamiltonian cases in [8]. Here we want to study variational equations and their Galois group near another type of particular orbit that we often encounter when the potential is invariant by rotation. In particular, if there exists a plane of Darboux points, invariant by the rotational symmetry of V (this case in not rare), then we can build particular orbits with non zero angular momentum. Then we get a one parameter family of orbits on which we can apply Morales Ramis theory. For all of them, the identity component of the Galois group of the variational equation should be abelian, and thus we can expect a much stronger integrability criterion than [74]. One difficulty is that the variational equation is intricate to study in the general case, thus we will make a complete analysis only in the case we will call "partially decoupled". We find very strong conditions, only two eigenvalues are possible instead of an infinity. Moreover we will see that this type of orbit allows us to study a new type of partial integrability: the case where the potential would be integrable only for a fixed value of the Hamiltonian and angular momentum.

The main theorem of this chapter is the following.

Theorem 32. *The n body problem with equal masses in the plane is neither meromorphically integrable on any hypersurface of the form $C^2H=\alpha$ with $\alpha\neq 0$ fixed, nor on the hypersuface $H=0$, nor on the hypersurface $C=0$ (H being the Hamiltonian, and C the total angular momentum).*

This result generalizes some already known non integrability proofs as [13] for $n=3, C=0$ (generalized in [47]), [55] for $n\geq 3, C=0$ and [68] for $n=3, H=0$ (later generalized in [69]). Along the proof of Theorem 32, we will prove more generally that if a homogeneous potential of degree -1 invariant by rotation is meromorphically integrable on a surface with fixed energy and "angular momentum" (as defined by equation (5.3)), then the eigenvalue λ of the Hessian matrix of V on a Darboux point with multiplier -1 should belong to the table 5.1 (which is found through the analysis of the Galois group of variational equations near conic orbits). The complete statement is Theorem 34, in which there is an additional a priori hypothesis, the "decoupling condition". Partially integrable potential exists effectively as given in (5.15). This integrability table also gives indications on which particular level of energy and "angular momentum" we should focus when searching partial integrability. These cases correspond also to regular confluences of the variational equations (even without the "decoupling condition"), and the Galois group being typically smaller for these cases, it is always worthwhile to consider them in particular. The case $C^2H=-1/2$ leads for the restricted 3 body problem to an additional first integral on this level (the Jacobi integral), although this first integral is not valid everywhere on $C^2H=-1/2$. For the general 3 body problem, this case will correspond to the energy and total angular momentum of circular motions on Lagrange and Euler central configurations. Theorem 34 cannot solve all problems of this kind because of this "decoupling condition". A complete analysis of the 3 body case gives all the masses which satisfy this condition in Proposition 15, which are not always symmetric. A non integrability theorem like Theorem 32 is by the way immediate for these masses, except for $(m_1,m_2,m_3)=(1,5,1)$.

5.2 General properties

Definition 22. *(See [29, 25] for an overview of interesting properties) We will call "norm" and scalar product the expressions*

$$\|v\|^2 = \sum_{i=1}^{n} v_i^2 \qquad < v, w >= \sum_{i=1}^{n} v_i w_i$$

even for complex v, w (in particular, the "norm" can vanish for non zero v). We will say moreover that a matrix is orthonormal complex if its columns $X_1, \dots, X_n$ are such that

$$< X_i, X_j >= \sum_{k=1}^{n} (X_i)_k (X_j)_k = 0 \quad \forall i, j \qquad \|X_i\|^2 = \sum_{k=1}^{n} (X_i)_k^2 = 1 \quad \forall i$$

We define $\mathbb{O}_n(\mathbb{C})$ the complex orthogonal group which is the group generated by these matrices, and $\mathbb{SO}_n(\mathbb{C})$ the subgroup of $\mathbb{O}_n(\mathbb{C})$ of matrices with determinant 1 (corresponding to rotations). In particular, the group $\mathbb{O}_n(\mathbb{C})$ preserves the "norm".

Definition 23. *Let V be a homogeneous meromorphic potential of degree -1 in dimension $n \geq 2$. We define*

$$G = \{g \in \mathbb{O}_n(\mathbb{C}), \;\; V(g.x) = V(x) \;\; \forall x \in \mathbb{C}^n\} \tag{5.2}$$

We will call G the symmetry group of V. For $v \in \mathbb{C}^n$, we define

$$Gv = \{\alpha g.v, \; \alpha \in \mathbb{C}, \; g \in G\}$$

We will say that V is invariant by rotation if G contains at least a subgroup isomorphic to $\mathbb{SO}_2(\mathbb{C})$. We will say that v is an eigenvector of G if for all $g \in G$, v is an eigenvector of g.

Note that we constraint variable change in q to be in the complex orthogonal group $\mathbb{O}_n(\mathbb{C})$ because this variable change also implies to do the same with the variables p in the Hamiltonian. The kinetic part is the "norm" of p is so is preserved by such transformation. Thus if $A \in \mathbb{O}_n(\mathbb{C})$ is in the symmetry group of V, we also get for the corresponding Hamiltonian $H(Ap, Aq) = H(p, q)$.

In the following, we will have to consider three notions of angular momentum. The first one (and the most general) is a (non constant) first integral of a potential V on $\mathbb{C}^n$ of the form

$$C = \sum_{0 \leq i < j \leq n} a_{i,j}(p_i q_j - p_j q_i) \qquad a_{i,j} \in \mathbb{C} \tag{5.3}$$

that we will call an "angular momentum" (with quotes). The second one is the canonical angular momentum on a plane $P \subset \mathbb{C}^n$ which equals to $p_2 q_1 - p_1 q_2$ for a direct orthonormal complex basis of P. Eventually, in the last part about the n body problem, we will consider the total angular momentum which is the sum of the canonical angular momenta of each body with respect to the center of mass.

The Noether Theorem (a simple statement can be found in [67]), makes a parallel between the symmetry group G of a potential V and the number of first integrals of V of the form (5.3). More precisely, the dimension of the Lie algebra of G equals to the dimension of the vector space of first integrals of the form (5.3).

Proposition 8. *Let V be a homogeneous potential of degree -1 in dimension $n \geq 2$ and G its symmetry group. Assume there exists a Darboux point c with $\|c\| \neq 0$ and $\tilde{G}$ a subgroup of G such that $P = \tilde{G}c$ is a plane. Then there exists a conic orbit on P and the variational equation near this conic orbit with parameters $(C\|c\|^2, H\|c\|^2) \in \mathbb{C}^2$ (canonical angular momentum on P and energy) is given by*

$$t(-C^2 + 2t + 2Ht^2)\ddot{X} + (-t + C^2)\dot{X} = R_{\theta(t)}^{-1} \nabla^2 V(c) R_{\theta(t)} X \tag{5.4}$$

where $R_{\theta(t)} \in \tilde{G}$ with coefficients in $\mathbb{C}\left(t, \sqrt{2H - C^2 t^{-2} + 2t^{-1}}\right)$

Proof. Let c be a Darboux point of V with multiplier -1 and $\tilde{G}$ a subgroup of G. After rotation of the coordinates, we can assume that $c = (\gamma, 0, \dots, 0)$ and that the plane $\tilde{G}c$ is generated by $(\gamma, 0, \dots, 0), (0, \gamma, 0, \dots, 0)$. A conic orbit for the Darboux point c corresponds to the orbit given by

$$(q_1, q_2) = \varphi_t(1, 0) \qquad q_i = 0 \quad i = 3 \dots n$$

where φ_t is given by

$$\varphi_t(x, y) = \phi(t) \begin{pmatrix} cos(\theta(t)) & -sin(\theta(t)) \\ sin(\theta(t)) & cos(\theta(t)) \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} \tag{5.5}$$

Replacing this in the Hamiltonian and canonical angular momentum (the potential V restricted to the plane P is invariant by rotation), we get

$$\frac{1}{2}\dot{\phi}^2\gamma^2 + \frac{1}{2}\gamma^2\phi^2\dot{\theta}^2 - \frac{V(c)}{\phi} = H\gamma^2 \qquad C\gamma^2 = \gamma^2\phi^2\dot{\theta}$$

And after replacing, we get

$$\frac{1}{2}\dot{\phi}^2\gamma^2 + \gamma^2\frac{C^2}{2\phi^2} - \frac{V(c)}{\phi} = H\gamma^2$$

Knowing that the multiplier is -1, we use Euler equation for V

$$\gamma^2 = V(c) \qquad \frac{1}{2}\dot{\phi}^2 + \frac{C^2}{2\phi^2} - \frac{1}{\phi} = H$$

Therefore, the variational equation is of the form

$$\ddot{X} = \frac{1}{\phi(t)^3}\nabla^2 V(R_{\theta(t)}c)X$$

with $R_{\theta(t)}$ a rotation matrix. We have the the potential restricted to the plane P is invariant by rotation and homogeneous of degree -1, and so the orbit corresponds to a Keplerian orbit. Thus the orbit is a conic which of the focus is at 0. We then get the relation

$$\phi(t) = \frac{p}{1 + e\cos(\theta)}$$

with p and e some parameters depending on C, H. We get that $\cos(\theta), \sin(\theta)$ are rational fractions in $\phi, \dot{\phi}$. Then, with variable change $\phi \longrightarrow t$ we get the following expression

$$t(-C^2 + 2t + 2Ht^2)\ddot{X} + (-t + C^2)\dot{X} = \nabla^2 V(R_{\theta(t)}c)X \tag{5.6}$$

where the dot correspond to the derivation in t. We know that the potential V is invariant by rotation. Then the matrix $\nabla^2 V(R_{\theta(t)}c)$ corresponds to $\nabla^2 V(c)$ after a basis change and gives

$$\nabla^2 V(R_{\theta(t)}c) = R_{\theta(t)}^{-1}\nabla^2 V(c)R_{\theta(t)} \tag{5.7}$$

Replacing this in (5.6) gives us the equation (5.4). □

Remark 14. *The main difficulty of this variational equation is that it does not decouple after basis change. Indeed, we can make a basis change with some matrix P, but this matrix P should commute with the rotations $R_{\theta(t)}$.*

Let us now give a proper definition of what we will call integrable on some level of first integrals, as given in Theorem 32.

Proposition 9. *Let H be a meromorphic Hamiltonian on an open set $U \subset \mathbb{C}^{2n}$. Let $I_1, \dots, I_k$ be meromorphic first integrals of H such that*

$$\{I_i, I_j\} = 0 \quad \forall i, j = 1 \dots k$$

where $\{\ ,\ \}$ is the Poisson bracket. We define the ring

$$\mathcal{O} = \left\{(p,q) \longrightarrow g(p,q,I_1(p,q),\dots,I_k(p,q)) \ \ g \text{ holomorphic on } U \times \mathbb{C}^k \neq 0\right\}$$

We assume that $< I_1, \dots, I_k >$ is a prime ideal on $\mathcal{O}$ and let $K = Frac(\mathcal{O}/ < I_1, \dots, I_k >)$ the corresponding fraction field. Then the following functions are well defined

- *For all $i = 1 \dots k$, the functions $\varphi_i : K \longrightarrow K,\ \ f \longrightarrow \{f, I_i\}$.*
- *The function*
$$\Psi : \left(\bigcap_{i=1}^{k} \varphi_i^{-1}(0) \right)^2 \longrightarrow K,\ \ f,g \longrightarrow \{f,g\}$$
- *The functions $K^{n-k} \longrightarrow K$ which associate to $f_1, \dots, f_{n-k}$ a sub determinant of size $n \times n$ of the Jacobian matrix (a matrix of size $2n \times n$) of $I_1, \dots, I_k,\ f_1, \dots, f_{n-k}$.*

Remark that the prime ideal condition is necessary to define the field of fractions, but also has a geometrical interpretation. We ask that the manifold $(I_1, \dots, I_k) = 0$ to be smooth (differentiable). If the ideal is not prime, then each prime component will be an invariant manifold and should be studied separately. It is not always possible as there exists example where some first integrals factorize but each factor is not a first integral.

Proof. Let us write a representant of $f \in K$ as $P/Q,\ P,Q \in \mathcal{O}$. We just need to check that the value of the function φ_i does not depend on the choice of the representant. We consider $h_1, \dots, h_k, g_1, \dots, g_k \in \mathcal{O}$ and we have

$$\left\{ \frac{P + \sum\limits_{s=1}^{k} h_s I_s}{Q + \sum\limits_{s=1}^{k} g_s I_s}, I_i \right\} =$$

$$\left(Q + \sum_{s=1}^{k} g_s I_s\right)^{-1} \left\{P + \sum_{s=1}^{k} h_s I_s, I_i\right\} - \frac{P + \sum\limits_{s=1}^{k} h_s I_s}{\left(Q + \sum\limits_{s=1}^{k} g_s I_s\right)^2} \left\{Q + \sum_{s=1}^{k} g_s I_s, I_i\right\} =$$

$$\left(Q + \sum_{s=1}^{k} g_s I_s\right)^{-1} \{P, I_i\} - \frac{P + \sum\limits_{s=1}^{k} h_s I_s}{\left(Q + \sum\limits_{s=1}^{k} g_s I_s\right)^2} \{Q, I_i\} =$$

$$Q^{-1}\{P, I_i\} - PQ^{-2}\{Q, I_i\} = \left\{\frac{P}{Q}, I_i\right\}$$

so the function is well defined on K.

Let us consider $f_1, f_2 \in \cap_{i=1}^{k} \varphi_i^{-1}(0)$ and we write P/Q a representant of f_1. Using the fact that $\{I_i, f_2\} = 0$, we can do exactly the same calculations as before just replacing $, I_i\}$ by $, f_2\}$. Using the fact that the Poisson bracket is symmetric, we can do the same interverting the indices $1, 2$. So the function Ψ is well defined.

Let us consider x one of the $2n$ variables $p, q,\ f \in K$ and P/Q a representant. We have the classical formula

$$\partial_x \left(\frac{P}{Q}\right) = Q^{-1}\partial_x P - PQ^{-2}\partial_x Q$$

So adding elements of $< I_1, \dots, I_k >$ to P and Q corresponds to add in the determinant a linear combination of $\partial_x I_1, \dots, \partial_x I_k$. The determinant being multilinear, this will not change the value of the determinant because it contains the columns of the derivatives of $I_i,\ i = 1 \dots, k$. □

One needs to be extremely cautious when manipulating these derivatives, because K is not a differential field, so we cannot conclude directly that all notions we will need (Poisson brackets, independence) are well defined. For example, the Jacobian matrix itself is not well defined on K, only its sub-determinants of size $n \times n$ are. Remark that, in the following, we will always consider a representant and will forget the field K, and so this complicated definition will have no impact.

Definition 24. *Let V be a homogeneous meromorphic potential of degree -1 in dimension $n \geq 2$. Let $I_1, \dots, I_k$ be meromorphic first integrals satisfying the hypotheses of Proposition 9. We say that V is meromorphically integrable on the manifold $(I_1, \dots, I_k) = 0$ (or partially integrable) if there exists $F_1, \dots, F_{n-k} \in K$ (K is defined as in Proposition 9) such that*

$$\{H, F_i\} = 0 \in K \ \ \forall i \qquad \{I_i, F_j\} = 0 \in K \ \ \forall i, j \qquad \{F_i, F_j\} = 0 \in K \ \ \forall i, j$$

and such that at least one of the subdeterminants of size $n \times n$ of the Jacobian matrix of $I_1, \dots, I_k, F_1, \dots, F_{n-k}$ is not 0 in K (this corresponds to the condition of independence almost everywhere).

Remark 15. *This definition of partial integrability implies the integrability of the differential system restricted to $\cap_{i=1}^k I_i^{-1}(0)$ in the Bogoyavlensky sense [11]. Indeed in the case of a partially integrable potential with definition 24, the restricted corresponding differential system is of dimension $2n - k$, there are $n - k$ first integrals F_i and n commutating vector fields given by $J\nabla I_i,\ i = 1 \dots k,\ J\nabla F_i,\ i = 1 \dots n - k$ (J is the matrix of the canonical symplectic form). Moreover, for this broader definition of integrability, we have the following generalization of Theorem 30.*

Theorem 33. *[8] Assume that a dynamical system given by a holomorphic vector field X on a complex analytic variety M is meromorphically integrable in the Bogoyavlensky sense, and let $\Gamma \subset M$ a non-stationary solution of X. Then for any natural number $n \geq 1$, the identity component of the differential Galois group of the variational equation of order n of X along Γ is Abelian.*

Proposition 10. *Let $V \neq 0$ be a homogeneous meromorphic potential of degree -1 in dimension $n \geq 2$ and assum there exists a non trivial first integral C of V of the form (5.3). Let us fix the value of the Hamiltonian $H = H_0 \neq 0$ and $C = C_0 \neq 0$. If V is integrable on this manifold of codimension 2, then V is integrable on the hypersurface $C^2 H = C_0^2 H_0$.*

Proof. We consider the following transformation

$$\varphi \ \ \mathbb{C}^{2n} \longrightarrow \mathbb{C}^{2n} \qquad (p, q) \longrightarrow (\alpha p, \alpha^{-2} q) \tag{5.8}$$

We see that the transformation φ just multiplies the Hamiltonian $H \longrightarrow \alpha^2 H$, and this does not change the integrability of H. Assume that H is integrable on the manifold $(H, C) = (H_0, C_0)$. We have

$$H(\varphi(p, q)) = \alpha^2 H \qquad C(\varphi(p, q)) = \alpha^{-1} C \qquad (C^2 H)(\varphi(p, q)) = C^2 H$$

Then H is also integrable on the manifold $(H, C) = (\alpha^2 H_0, \alpha^{-1} C_0)$. We also have

$$\bigcup_{\alpha \in \mathbb{C}^*} \left(H^{-1}(\alpha^2 H_0) \cap C^{-1}(\alpha^{-1} C_0)\right) = \left\{(p, q) \in \mathbb{C}^{2n},\ \ C(p, q)^2 H(p, q) = C_0^2 H_0\right\}$$

because $C_0^2 H_0 \neq 0$. This gives the theorem. □

Remark 16. *We can see that the study of integrability on a specific manifold makes sense only if this manifold is invariant by* φ*, because if it is not the case, then our potential will be integrable on a manifold with higher dimension. Remark that the ideals* $< C-C_0, H-H_0 >$, $< C^2H-C_0^2H_0 >$ *are always prime for* $C_0^2H_0 \neq 0$ *and* $V \neq 0$*, so integrability on these manifolds is well defined, contrary to the case* $< C^2H >$ *which will need to be split in two parts* $< C >, < H >$*.*

5.3 Integrability table

Theorem 34. *Let* V *be a homogeneous meromorphic potential of degree* -1 *in dimension* $n \geq 2$ *and* G *its symmetry group. We consider* c *a Darboux point of* V *with* $\|c\| \neq 0$ *and multiplier* -1*, and* E *a subspace of an eigenspace of* $\nabla^2 V(c)$*. Assume there exists a subgroup* $\tilde{G}$ *of* G *such that* $P = \tilde{G}c$ *is a plane and that* E *is invariant by* $\tilde{G}$*. Considering an "angular momentum" whose restriction to* P *is the canonical angular momentum on* P*, if* V *is meromorphically integrable (respectively on some specific level* $(H_0, C_0) \in \mathbb{C}^2$ *of the Hamiltonian and "angular momentum"), then the following equation has a Galois group whose identity component is abelian (respectively for the parameters* $(H, C) = \|c\|^{-2}(H_0, C_0)$*)*

$$t(-C^2 + 2t + 2Ht^2)\ddot{X} + (-t + C^2)\dot{X} = \lambda X \qquad H, C, \lambda \in \mathbb{C} \tag{5.9}$$

where λ *is the eigenvalue of* $\nabla^2 V(c)$ *associated to the eigenspace* E*.*

Proof. This is a direct application of Proposition 8. We have a plane $P = \tilde{G}c$ and all vectors in this plane are Darboux points. The potential restricted to this plane is invariant by rotation (because $\tilde{G}$ is a subgroup of the symmetry group of V). On the eigenspace E, the matrix $\nabla^2 V(c)$ equals to λId_E. Moreover we know that the space E is invariant by the rotations $R_{\theta(t)}$ which correspond to elements of $\tilde{G}$. Thus we have

$$R_{\theta(t)}^{-1} \nabla^2 V(c) R_{\theta(t)} \Big|_E = \lambda Id_E$$

So the equation (5.4) on the eigenspace E simplifies and becomes equation (5.9). The condition on the Galois group of equation (5.9) comes either from Theorem 30 in the case of complete integrability, or from Theorem 33 in the partially integrable case (in this case X is the Hamiltonian vector field restricted to the level (H_0, C_0), and the manifold M is an open neighbourhood of the conic orbit on this level). □

Remark 17. *The Theorem 34 has lots of hypotheses, but in fact only one of them is really restrictive. The existence of an invariant plane* $\tilde{G}c$ *on which the potential is invariant by rotation is common in practical cases. This often results from the symmetry of the system. This is for example always the case in the* n *body problem. The restrictive condition is the existence of* E *invariant by* $\tilde{G}$*. In fact, this is a condition very similar to the codiagonalization constraint that Maciejewski-Przybylska found when studying potentials which are the sum of two homogeneous potentials. In fact, a potential invariant by rotation in dimension* n *can also be reduced to become a potential in dimension* $n-1$ *which will be a sum of a homogeneous potential and the potential* C^2/r^2*. This new potential is not homogeneous and our condition corresponds to the commutation of the Hessian matrices (at least on some non trivial subspace).*

Lemma 27. *The differential equation* (5.9) *is a Fuchsian equation with* 4 *singularities, of Heun type [3]. The Galois group is* $SL_2(\mathbb{C})$ *except if the values of* (C, H, λ) *belong to table 5.1.*

Proof. We first remark that in the case $H = C = 0$, equation (5.9) simplifies to

$$2t^2\ddot{X} - t\dot{X} = \lambda X$$

which always has an abelian Galois group. Thus from now we will assume $(H,C) \neq (0,0)$. Using a linear variable change $t \longrightarrow \alpha t$ in equation (5.9), we get the following equation

$$t(2H\alpha^2t^2+2\alpha t-C^2)\ddot{X}-(\alpha t-C^2)\dot{X}=\alpha\lambda X \tag{5.10}$$

For $C^2H\neq 0$, we can choose $\alpha=(\sqrt{1+2C^2H}-1)/(2H)$ which gives the equation

$$\begin{gathered}2t(t-1)\left(\left(1-\sqrt{1+2C^2H}+C^2H\right)t+C^2H\right)\ddot{X}+\\ \left(\left(1-\sqrt{1+2C^2H}\right)t+2C^2H\right)\dot{X}=\left(-1+\sqrt{1+2C^2H}\right)\lambda X\end{gathered} \tag{5.11}$$

We begin by the case $C^2H\neq -1/2$. The equation (5.11) has exactly 4 regular singularities on

$$0,1,\frac{C^2H}{\gamma-1-C^2H},\infty$$

where $\gamma=\sqrt{1+2C^2H}$. We make Frobenius expansion on these 4 singularities, and we find a logarithmic term for $t=0$ and for $t=\infty$. More precisely, we get

$$\begin{aligned}X(t)=c_1t^2\left(1-\frac{(\lambda-2)(\gamma-1)}{6C^2H}t+O(t^2)\right)+\\ c_2\left(\ln t\left(\frac{\lambda(1+\lambda)(1-\gamma+C^2H)}{2C^4H^2}t^2+O\left(t^3\right)\right)-2-\frac{(\gamma-1)\lambda}{C^2H}t+O(t^2)\right)\\ X(t)=c_1\left(1+\frac{(\gamma-1)\lambda}{4(1-\gamma+C^2H)t}+O\left(\frac{1}{t^2}\right)\right)+\\ c_2\left(\ln t\left(\frac{(1+\lambda)(\gamma-1)}{2(1-\gamma+C^2H)}+O\left(\frac{1}{t}\right)\right)-t\left(1-\frac{(\gamma-1)}{2(1-\gamma+C^2H)t}+O\left(\frac{1}{t^2}\right)\right)\right)\end{aligned}$$

These expansions are valid for $\lambda\neq -1,0$. In the case $\lambda=-1$, we can compute explicitly the solutions and we find

$$X(t)=c_1\left(t-\gamma-1\right)+c_2\sqrt{\left(t-1\right)\left(\gamma+1+C^2H(t+1)\right)}$$

The Galois group is then $\mathbb{Z}/2\mathbb{Z}$, abelian. In the case $\lambda=0$, we find the solution

$$\begin{aligned}X(t)=c_1+c_2\sqrt{(t-1)(2t-2t\gamma+2C^2H(t+1))}+\\ c_2\ln\left(-1-t\gamma+t+\sqrt{(t-1)(2t-2t\gamma+2C^2H(t+1))}\right)\end{aligned}$$

The identity component of the Galois group is then $\mathbb{C}$, thus abelian. Let us consider the case $\lambda\neq -1,0$. Among the three solvable cases of Kovacic's algorithm [41], the only possible one with a logarithmic term requires the existence of a solution of the form

$$X(t)=\exp\left(\int F(s)ds\right)\qquad F\in\mathbb{C}(t)$$

If F has singularities of order more than 2 then X does not have a Puiseux expansion near this singularity. This is impossible because all singularities are regular. If the degree of F is positive, then the expansion at infinity is not a Puiseux series. Then the particular solution $X(t)$ should be of the following form

$$X(t)=\prod_{i=1}^{k}(t-t_i)^{m_i}$$

If m_i is not a non-negative integer, then t_i is a singularity of X and equals to one of the singularities of the equation. This gives even more constraints on the m_i because the Frobenuis exponents on $1, C^2H/(\gamma-1-C^2H)$ are $0, 1/2$. On 0, the possible exponent is 2, and on infinity it is 0 (the other ones correspond to the logarithmic behavior). This implies that the sum of the m_i is zero. The m_i being all non-negative, all of them are zero. The only left possibility is then $X(t)=1$. We replace in equation (5.11) and we find $\lambda=0$, case already done. Then the Galois group is $SL_2(\mathbb{C})$.

The cases $C=0, H=0, C^2H=-1/2$ correspond to confluences. These confluences are all regular (this has probably something to do with the fact that the system comes from a variational equation of a Hamiltonian system). The case $C=0$ has already been treated by [74, 53]. Let us study the case $H=0$. This corresponds to the parabolic case (some study of this case has already been done by [20]). Putting $\alpha=C^2/2$, equation (5.10) becomes

$$2t(t-1)\ddot{X}-(t-2)\dot{X}=\lambda X$$

There is a logarithmic term for the singularity $t=0$

$$X(t)=c_1t^2\left(1+\left(\frac{1}{3}-\frac{1}{6}\lambda\right)t+O(t^2)\right)+$$
$$c_2\left(ln\, t\left(\left(\frac{1}{4}\lambda^2+\frac{1}{4}\lambda\right)t^2+O(t^3)\right)-2-\lambda t-\left(\frac{1}{2}\lambda+\frac{1}{4}\right)t^2+O(t^3)\right)$$

for $\lambda\neq 0,-1$. In the cases $\lambda\in\{0,-1\}$, we find the solutions

$$X(t)=c_1+c_2\sqrt{t-1}(2+t) \qquad X(t)=c_1(t-2)+c_2\sqrt{t-1}$$

The Galois group is then $\mathbb{Z}/2\mathbb{Z}$ in both cases, then abelian. We now look at the case $\lambda\neq 0,-1$. The possibles exponents are $\{2\}$ at 0, $\{0,1/2\}$ at 1 and

$$\left\{-\frac{3}{4}+\frac{1}{4}\sqrt{8\lambda+9},-\frac{3}{4}-\frac{1}{4}\sqrt{8\lambda+9}\right\}$$

at ∞. As before, we prove that we need a solution of the form

$$X(t)=\prod_{i=1}^{k}(t-t_i)^{m_i}$$

All possible exponents outside infinity are integers or half integers, non-negative, thus the sum of the m_i is a non-negative integer or half integer. Then

$$-\frac{3}{4}+\frac{1}{4}\sqrt{8\lambda+9}=\frac{1}{2}(k-1) \qquad k\in\mathbb{N}^*$$

We solve this equation and we find

$$\lambda=\frac{1}{2}(k-1)(k+2) \qquad k\in\mathbb{N}^*$$

This is exactly the condition of Lemma 27. We now want to compute the Galois group for these remaining cases. We write the solutions of the equation in the following form (it is a hypergeometric equation, and the solutions can be written using hypergeometric series ${}_2F_1$)

$$X(t)=c_1\,{}_2F_1\left(\left[1-\frac{1}{2}k,\frac{1}{2}k+\frac{3}{2}\right],\left[\frac{1}{2}\right],-t+1\right)t^2+$$
$$c_2\,{}_2F_1\left(\left[2+\frac{1}{2}k,\frac{3}{2}-\frac{1}{2}k\right],\left[\frac{3}{2}\right],-t+1\right)\sqrt{t-1}t^2$$

These hypergeometric series are finite if the first bracket in ${}_2F_1$ contains a non-positive integer. For $k \geq 2$, we see that either $1 - \frac{1}{2}k$ or $\frac{3}{2} - \frac{1}{2}k$ is a non-positive integer. Then one of the two functions is a polynomial. We always have a solution in $\mathbb{C}[t, \sqrt{t-1}]$, and then the identity component of the Galois group is abelian.

Let us now study the case $C^2H = -1/2$. The equation (5.11) becomes

$$-t(t-1)^2\ddot{X} - (t-1)\dot{X} = \lambda X$$

The expansion on 0 is the following

$$X(t) = c_1 t^2 \left(1 + \left(\frac{1}{3} - \frac{1}{6}\lambda\right) t + \left(\frac{1}{96}\lambda^2 - \frac{11}{96}\lambda + \frac{3}{16}\right) t^2 + O(t^3)\right) +$$
$$c_2 \left(\ln\, t \left(\left(\frac{1}{4}\lambda^2 + \frac{1}{4}\lambda\right) t^2 + O(t^3)\right) - 2 - \lambda t - \left(\frac{1}{2}\lambda + \frac{1}{4}\right) t^2 + O(t^3)\right)$$

and has a logarithmic term for $\lambda \neq 0, -1$. The expansion at infinity is

$$X(t) = c_1 \left(1 - \frac{\lambda}{2t} + \frac{\lambda(\lambda-5)}{12t^2} + O(t^{-3})\right) +$$
$$c_2 \left(\ln\, t \left(\lambda + 1 - \frac{\lambda(\lambda+1)}{2t} + O(t^{-2})\right) + t + 1 - \frac{4 + 11\lambda + 3\lambda^2}{4t} + O(t^{-2})\right)$$

Then there is always at least one logarithmic term for $\lambda \neq -1$. Remark that we already know that this equation has an abelian Galois group for $\lambda = 0, -1$ (either using the limiting process of the generic solution for all C, or running Kovacic's algorithm for these specific cases). So from we assume that $\lambda \neq 0, -1$. We know that if the Galois group is not $SL_2(\mathbb{C})$, then there exists a solution of the form

$$X(t) = \exp\left(\int F(s)ds\right) \qquad F \in \mathbb{C}(t)$$

The equation is Fuchsian and then $X(t)$ can be written

$$X(t) = \prod_{i=1}^{k} (t - t_i)^{m_i}$$

The m_i need to be non-negative integers except maybe at singularities. The exponents at 1 are $+\sqrt{-\lambda}, -\sqrt{-\lambda}$. Then one of the following equation is satisfied

$$2 + \sqrt{-\lambda} + k = 0 \quad \text{or} \quad 2 - \sqrt{-\lambda} + k = 0 \qquad k \in \mathbb{N}$$

Then

$$\lambda = -(k+2)^2 \quad k \in \mathbb{N}$$

We add the cases $\lambda = 0, -1$ and this gives exactly the condition given by Lemma 27. We now need to compute the Galois group for these specific cases. We write the solutions of the equation in the following form (it is a hypergeometric equation, and the solutions can be written using hypergeometric series)

$$X(t) = c_1\, {}_2F_1\left(\left[2 - i\sqrt{\lambda}, 1 - i\sqrt{\lambda}\right], \left[1 - 2i\sqrt{\lambda}\right], 1 - t\right) t^2 (t-1)^{-i\sqrt{\lambda}} +$$
$$c_2\, {}_2F_1\left(\left[1 + i\sqrt{\lambda}, 2 + i\sqrt{\lambda}\right], \left[1 + 2i\sqrt{\lambda}\right], 1 - t\right) t^2 (t-1)^{i\sqrt{\lambda}}$$

These hypergeometric series are finite if the first bracket in the hypergeometric series ${}_2F_1$ contains a non-positive integer. We see that for $\lambda = -k^2 \quad k \in \mathbb{N}^*$, it is the case for the solution in c_1. There is always a polynomial solution and then the Galois group is always abelian. □

Remark that such a work can also be done using the classification of hypergeometric equation which are solvable by quadrature in [37].

5.4 Algebraic potentials

In the following sections, we will often need to consider algebraic potentials instead of meromorphic ones. This is a problem because Theorem 31 deals only with meromorphic potentials. This problem is often not addressed, except in [77], but in fact his procedure does not work. This is because making cuts in the complex plane does not allow afterwards to make all possible monodromy paths. Then, the monodromy group will be reduced. It could have no consequences, but here there are important consequences because we absolutely need to be able to turn around the point 0 in the variational equation (this is because for the two other singularities, the exponents are $0, 1/2$, thus if we restrict ourselves to these ones, the monodromy group will always be abelian). This problem has been raised by Theorem 2. of Combot [21].

We consider polynomials $G_1, \dots, G_s \in \mathbb{C}[q_1, \dots, q_n, w_1, \dots, w_s]$ and the ideal $I = < G_1, \dots, G_s >$. We assume that I is a prime ideal and that the matrix

$$J \in M_s(\mathbb{C}[q, w]) \qquad J_{i,j} = \frac{\partial G_i}{\partial w_j}, \quad i, j = 1 \dots s$$

has a non-zero determinant modulo the ideal I. We define the associated manifold $\mathcal{S} = I^{-1}(0)$. For a potential on $\mathcal{S}$, we define the derivations

$$\frac{\partial V}{\partial q_k} = \partial_k V - (\partial_{n+1} V, \dots, \partial_{n+s} f)\, J^{-1}\, (\partial_k G_1, \dots, \partial_k G_s)^\intercal \tag{5.12}$$

This formula is well defined when J is invertible and outside the singularities of V. This singular set corresponds to

$$\Sigma(V) = \{(q, w) \in \mathcal{S},\ \ V \text{ is not } C^\infty \text{ at } (q, w)\}$$

when the derivability in respect to the q_i as in (5.12). The dynamical system associated to V is the following

$$\dot{q}_i = p_i, \quad \dot{p}_i = -\frac{\partial V}{\partial q_i}, \quad i = 1 \dots n \qquad \dot{w}_i = \sum_{j=1}^{s} p_j \frac{\partial w_i}{\partial q_j}, \quad i = 1 \dots s \tag{5.13}$$

Now that we have well defined the dynamical system corresponding to a potential on an algebraic manifold, we can prove a Theorem similar to 34

Theorem 35. *Let V be a homogeneous meromorphic potential of degree -1 on an open set U of a complex algebraic manifold $\mathcal{S}$ of dimension $n \geq 2$ and G its symmetry group. We consider $c \in U \setminus \Sigma(V)$ a Darboux point of V with $\|c\| \neq 0$ and multiplier -1, and E a subspace of an eigenspace of $\nabla^2 V(c)$. Assume there exists a subgroup $\tilde{G}$ of G such that $P = \tilde{G}c$ is a 2-dimensional cone*

$$\mathcal{C} = \{(x_1, x_2, r) \in \mathbb{C}^3,\ x_1^2 + x_2^2 = r^2\}$$

and that E is invariant by $\tilde{G}$. Considering an "angular momentum" whose restriction to P is the canonical angular momentum on P, if V is integrable (respectively on some specific level $(H_0, C_0) \in \mathbb{C}^2$ of the Hamiltonian and "angular momentum") with first integrals meromorphic on $\mathbb{C}^n \times U$, then equation (5.9) has a Galois group whose identity component is abelian (respectively for the parameters $(H, C) = \|c\|^{-2}(H_0, C_0)$), where λ is the eigenvalue of $\nabla^2 V(c)$ associated to the eigenspace E.

Introducing algebraic complex varieties $\mathcal{S}$ and $\mathcal{C}$ are necessary to analyze potentials containing algebraic extensions as the n body problem (in this case, the algebraic extensions w_i will be the mutual distances). The manifold $\mathcal{S}$ is the manifold on which the potential is properly defined. The variety $\mathcal{C}$ is associated to the planar potential $V = (x_1^2 + x_2^2)^{-1/2}$, which is the unique (up to a factor) planar homogeneous potential of degree -1 invariant by rotation. This means by the way that in Theorem 34, an invariant plane P on which a meromorphic homogeneous potential of degree -1 is invariant by rotation is a trivial one, as the restriction to P of V should be of the form $a(x_1^2+x_2^2)^{-1/2}$, and then non-meromorphic on $\mathbb{C}^2$ unless $a=0$.

Proof. This is almost the statement of Theorem 34, and the only difference is that we consider V on an open set U of complex manifold $\mathcal{S}$. We follow the proof of Theorem 2. of Combot [21]. We know that V is homogeneous and invariant by $\tilde{G}$, then so is the critical set $\Sigma(V)\subset\mathcal{S}$. Therefore, if $c\notin\Sigma(V)$, then the whole conic orbit is not in $\Sigma(V)$ except maybe for $q=0$. We define Γ the algebraic curve corresponding to the conic orbit (without the singular point $q=0$), and $M\subset\mathbb{C}^n\times U$ an open neighbourhood of Γ on which the potential is holomorphic (recall that for each $x\in\mathcal{S}\setminus\Sigma(V)$, V is holomorphic on a neighbourhood of x). The first integrals of V are meromorphic on $\mathbb{C}^n\times U$, and so are meromorphic on M. We can then apply Theorem 30 with the manifold M (respectively Theorem 33 with X the Hamiltonian vector field of H restricted to the level (H_0,C_0) and M restricted to this level). Following the proof of Theorem 34, equation (5.9) is the variational equation near Γ restricted to the eigenspace E, and so has a virtually abelian Galois group.

To conclude, we now need to precise that the Galois group in Theorems 30, 33 is over the field of meromorphic functions on Γ, which corresponds after variable change to meromorphic functions in $t,\sqrt{2E-C^2t^{-2}+2t^{-1}},\quad t\neq 0$. As Morales-Ramis using Kimura table [37] have done, we are in fact computing the Galois group of (5.9) over the base field $\mathbb{C}(t)$. Still the variational equation (5.9) being Fuchsian, we can apply Lemma 3. of Combot [21], and then the Galois group over the base field $\mathbb{C}(t)$ and the one over the base field of meromorphic function in $t,\ t\neq 0$ are equal. Thus the Galois group of (5.9) over the base field of meromorphic functions in $t,\sqrt{2E-C^2t^{-2}+2t^{-1}},\quad t\neq 0$ is at most an extension of degree 2 of the one over the base field $\mathbb{C}(t)$, and so the identity component will be the same. ☐

5.5 The case of dimension 3

In the particular case of dimension 3, we get

Proposition 11. *We consider the ideal*

$$I=\left\langle w_1^2-x^2-y^2,w_2^2-x^2-y^2-z^2\right\rangle$$

and V a potential meromorphic in w_1,w_2,z^2 on $I^{-1}(0)$ and homogeneous of degree -1 (this implies that the symmetry group of V contains $\mathbb{Z}/2\mathbb{Z}\times\mathbb{O}_2$). Assume that $V(1,0,0)\neq 0,\infty$. If V is meromorphically integrable, then it belongs to one of the following families

$$V=\frac{a}{w_2}\quad a\in\mathbb{C}^*\qquad V=\frac{b}{w_1}\quad b\in\mathbb{C}^* \tag{5.14}$$

This theorem is almost the best we can have (with a reasonable statement). To apply our previous theory, we need an invariant plane on which the potential is invariant by rotation. Such an invariant plane comes here from the symmetry in z. The constraint $V(1,0,0)\neq\infty$ cannot be removed, but the constraint $V(1,0,0)\neq 0$ could maybe be removed with a lot of additional work. There are two keys which allow us to give such a complete statement, which are the fact that the decoupling condition is always satisfied, and then that the potential can be reduced on a plane for which an almost complete classification is already done in Combot [19] (for a finite number of eigenvalues).

Proof. The potential V admits a symmetry group G such that

$$G\supset\left\langle\begin{pmatrix}cos(\theta)&-sin(\theta)&0\\sin(\theta)&cos(\theta)&0\\0&0&1\end{pmatrix},\begin{pmatrix}1&0&0\\0&1&0\\0&0&-1\end{pmatrix}\right\rangle$$

We consider P the plane $z=0$. This is an invariant plane because $\partial_z V(x,y,0)=0$ thanks to parity in z. Using the hypotheses, the restriction of V to the plane P is not zero or infinite. Therefore the point $c=(1,0,0,1,1)$ is a non degenerate Darboux point. The matrix $\nabla^2 V(c)$ contains a stable subspace of dimension 2 associated to P. Then the supplementary space generated by the vector $(0,0,1)$ is also an eigenspace. The rotation group generated by the rotations around the z-axis let the vector $(0,0,1)$ invariant. The conditions of Theorem 35 are satisfied and the "vertical" (normal to the plane P) variational equation is then

$$t(-C^2+2t+2Et^2)\ddot{X}+(-t+C^2)\dot{X}=\partial_{zz}V(c)X$$

This equation is integrable for all values of C only if

$$\partial_{zz}V(c)\in\{0,-1\}$$

with c with multiplier equal to -1. We now restrict our potential to the plane $\tilde{P}:\ \ y=0$. The potential V is invariant by rotation around the z-axis, then $\tilde{P}$ is an invariant plane and we consider the restriction $\tilde{V}:\tilde{P}\mapsto\bar{\mathbb{C}}$.
The restriction of w_1 to $y=0$ leads to two choices which are are $+x$ or $-x$. We can choose either for the restriction $\tilde{V}$ (because $\tilde{V}$ should be integrable for both possibilities anyway), and we choose arbitrary

$$w_1|_{y=0}=x$$

The function $\tilde{V}(x,z)$ is then meromorphic in x,z,w_3 on the complex manifold $\tilde{I}^{-1}(0)$ where $\tilde{I}=<w_3^2-x^2-z^2>$. It has a Darboux point $c=(1,0,1)$, and it is non degenerate. There are only two possible eigenvalues for the Hessian matrix on this Darboux point, and so we have a complete classification of such integrable potentials in Combot [19]. Then, applying the symmetry group, we find that if V is meromorphically integrable, then it should be of the form (5.14). These potentials effectively has an additional first integral, respectively p_zx-p_xz and p_z, which are functionally independent with the Hamiltonian and the canonical angular momentum p_xy-p_yx. □

Remark 18. *We can see here the importance of the symmetry group structure in the study of integrability. Here, the vertical variational equation is simple because in dimension 3, a group of rotations (except $\mathbb{SO}_3$) always admits a common eigenvector. This is no more the case in dimension 4 and higher. In particular, the complexity of the variational equation is closely linked to the symmetry group, and if it is too complicated, we will need additional properties on the matrix $\nabla^2 V(c)$. As we will see afterwards, in the n body problem, an explicit decoupling condition appear because the symmetry group contains the rotations*

$$R_\theta=\begin{pmatrix}\cos\theta I_n & -\sin\theta I_n\\ \sin\theta I_n & \cos\theta I_n\end{pmatrix}$$

and this group does not admits a common eigenvector of eigenvalue 1.

Now we can wonder whether the Theorems 34, 35 are really "useful" and not only purely theoretical possibilities without any examples. Does there exist effectively some potentials that would be integrable only for a specific energy and value of an "angular momentum"? Using Hietarinta's [31] direct method and then our non-integrability approach, we find the following potentials (the second one probably being new).

Proposition 12. *We consider the potentials*

$$V_1=\frac{w_1}{x^2+y^2-z^2}\qquad V_2=\frac{x^2+y^2+z^2}{w_1^3}\qquad I=\left\langle w_1^2-x^2-y^2\right\rangle \tag{5.15}$$

The potential V_1 is integrable for zero canonical angular momentum $C=p_xy-p_yx=0$, but not on any other hypersurface $C^2H=\alpha,\ \ \alpha\in\mathbb{C}^*$ (the question about integrability on $H=0$ is still

open).
The potential V_2 is integrable on the hypersurface $H = 0$ of zero energy, but neither on any other hypersurface $C^2 H = \alpha,\ \ \alpha \in \mathbb{C}^*$, nor on the hypersurface $C = 0$.

Proof. The first integral of V_1 is given by

$$I_1 = \frac{(xp_x + yp_y)p_z}{w_1} - \frac{z}{x^2 + y^2 - z^2}$$

The potential V_1 has a Darboux point $(1, 0, 0)$, and the associated eigenvalue is $\lambda = 2$. Using the integrability table 5.1, this value is the only possible one for the hypersurfaces $C = 0$ and $H = 0$. Then V_1 is not meromorphically integrable on any hypersurface of the form $C^2 H = \alpha,\ \ \alpha \in \mathbb{C}^*$. The first integral of V_2 is given by

$$I_2 = (x^2 + y^2 - z^2)^2 p_z^2 - 4z(x^2 + y^2 - z^2)p_z(xp_x + yp_y) + 4z^2(xp_x + yp_y)^2$$

The potential V_2 has a Darboux point $(1, 0, 0)$, and the associated eigenvalue is $\lambda = 2$. Using the integrability table 5.1, this value is the only possible one for the hypersurfaces $C = 0$ and $H = 0$. We know it is integrable for $H = 0$. Assume it is integrable for $C = 0$. Then, we could reduce the potential by rotation and we would obtain the following potential (on the plane $y = 0$)

$$\tilde{V_2} = \frac{x^2 + z^2}{x^3}$$

This potential has a Darboux point $(1, 0)$ and the associated eigenvalue is $\lambda = 2$. But in this case, it already has been proved in Combot [19] that the potential should belong to one of the following families (after rotation)

$$V = \frac{a}{x} + \frac{b}{z} \quad a, b \in \mathbb{C}^* \quad V = \frac{a(x^2 + z^2)}{(x + \epsilon iz)^3} + \frac{a}{x + \epsilon iz} \quad a \in \mathbb{C}^*,\ \epsilon = \pm 1 \tag{5.16}$$

The second case is impossible because it is always complex. For the first one, we apply a rotation to $\tilde{V_2}$ of angle θ

$$\tilde{V}_{2\theta} = \frac{x^2 + z^2}{\left(\cos(\theta)x + \sin(\theta)z\right)^3}$$

and this never coincide with expression (5.16). Then V_2 is not integrable on the hypersurface $C = 0$. ∎

5.6 Application to the n body problem

We consider V the potential of the n body problem in the plane

$$V = \sum_{i>j} \frac{m_i m_j}{r_{i,j}} \qquad I = \left\langle \left(r_{i,j}^2 - \|q_i\|^2\right)_{\substack{i,j=1\dots n \\ i>j}} \right\rangle \tag{5.17}$$

with positive masses m_i, $q_i \in \mathbb{C}^2$. The symmetry group is (at least)

$$G = \left\langle \begin{pmatrix} \cos\theta I_n & -\sin\theta I_n \\ \sin\theta I_n & \cos\theta I_n \end{pmatrix},\ \ \theta \in \mathbb{C} \right\rangle$$

Let c be a Darboux point with multiplier -1 and such that $\|c\|^2 \neq 0$. Then Gc is a plane and inside this plane we can build a conic orbit (by definition, the mutual distances between the bodies are not zero). For the following, we will pose

$$W_{i,j} = \frac{1}{m_i} \left(\nabla^2 V(c)\right)_{i,j} \qquad W \in M_{2n}(\mathbb{C}) \tag{5.18}$$

using notation $m_{i+n} = m_i$. Remark for the following that the potential of the n body problem as given by (5.17) is not reduced at all. This means in particular that the kinetic part is

$$\sum_{i=1}^{n} \frac{\|p_i\|^2}{2m_i}$$

and so does not correspond exactly to the case we studied before. Still, it is almost the same and we just have to make a variable change $p_i \longrightarrow p_i\sqrt{m_i}$. The matrix $\nabla^2 V(c)$ becomes notably the matrix given by (5.18).

5.6.1 General properties

Definition 25. *Let V be the potential of the n body problem with positive masses m_i, c a Darboux point with multiplier -1. We will say that the variational equation near a conic orbit is partially decoupled if there exists a non trivial vector space $\tilde{V}$ and $\lambda \in \mathbb{C}$ such that*

$$Wv = \lambda v \quad \forall v \in \tilde{V}$$

and $\tilde{V}$ is stable by the rotations

$$R_\theta = \begin{pmatrix} \cos\theta I_n & -\sin\theta I_n \\ \sin\theta I_n & \cos\theta I_n \end{pmatrix}$$

Remark 19. *This definition corresponds exactly to the existence of a non trivial eigenspace E satisfying Theorem 35.*

Lemma 28. *Let V be the potential of the n body problem with positive masses m_i, c a Darboux point with multiplier -1 and $W \in M_{2n}(\mathbb{C})$ the associated matrix (given by equation (5.18)). The variational equation near a conic orbit is partially decoupled if and only if there exists a vector $v \in \mathbb{C}^{2n} \setminus \{0\}$ and $\lambda \in \mathbb{C}$ such that*

$$Wv = J^{-1}WJv = \lambda v \tag{5.19}$$

where $J \in M_{2n}(\mathbb{C})$ is matrix of the canonical symplectic form.

Proof. Assume at first that v is not an eigenvector of R_θ (these matrices commute so they have the same eigenvectors). We just have to take $\tilde{V} = \operatorname{Span}(v, Jv)$ because the space generated by $R_\theta v$, $\forall\theta$ is a 2-dimensional space which contains v, Jv ($\tilde{V}$ is always 2-dimensional because v is not an eigenvector of $J = R_{\pi/2}$). Using the hypotheses, v and Jv are eigenvectors of W with the same eigenvalue, so $\tilde{V}$ is an eigenspace of W stable by the rotations R_θ. If v is an eigenvector of R_θ, then we take $\tilde{V} = \mathbb{C}.v$ and $\tilde{V}$ is an eigenspace of W stable by the rotations R_θ.
Conversely, if we have an eigenspace $\tilde{V}$ stable by the rotations R_θ, we take any vector $v \in \tilde{V}$ and it satisfies (5.19) because $J = R_{\pi/2}$ and then $Jv \in \tilde{V}$, and so it is likewise an eigenvector of eigenvalue λ. ☐

Theorem 36. *Let V be the potential of the n body problem with positive masses m_i, c a Darboux point with multiplier -1 and $W \in M_{2n}(\mathbb{C})$ the associated matrix (given by equation (5.18)). If the variational equation near a conic orbit is partially decoupled then the matrix W of (5.18) has a double eigenvalue.*

Proof. If $dim(\tilde{V}) \geq 2$ then by definition the matrix W has a double eigenvalue. Let us consider the case $dim(\tilde{V}) = 1$. The corresponding vector has to be a common eigenvector of J and W. The eigenvectors J are of the form (w, iw), $w \in \mathbb{C}^n$. In particular, they have zero "norm". But if W has only simple eigenvalues, then it is diagonalizable, so diagonalizable in an orthonormal complex basis (this is a small linear algebra proof done in [25] page 8 Theorem 4). So if W has an eigenvector with zero "norm", then this eigenvector is a linear combination of two eigenvectors and this implies the existence of an eigenspace of dimension ≥ 2, so a double eigenvalue. ☐

Proposition 13. *Let V be the potential of the n body problem with positive masses m_i, c a Darboux point with multiplier -1 **such that the bodies are aligned** and $W \in M_{2n}(\mathbb{C})$ the associated matrix (given by equation (5.18)). We assume that W is diagonalizable. Then the variational equation near a conic orbit has a Galois group G such that*

$$G \sim \tilde{G} \text{ with } \tilde{G} \subset \mathbb{C} \times Sp(2)^{n-2}$$

where $Sp(2)$ is the 4 dimensional symplectic group.

Proof. For an **aligned** Darboux point, we have the following property (found by direct computation)

$$W = \begin{pmatrix} A & 0 \\ 0 & -\frac{1}{2}A \end{pmatrix} \qquad J^{-1}WJ = \begin{pmatrix} -\frac{1}{2}A & 0 \\ 0 & A \end{pmatrix} \tag{5.20}$$

Then W and $J^{-1}WJ$ commute. Then there exists a common eigenvector basis of W and $J^{-1}WJ$. Then there exists a decomposition of $\mathbb{C}^{2n}$ in spaces V_i of dimension 2 with the V_i stable by rotations R_θ. We can then write the variational equation under the following form

$$t(-C^2 + 2t + 2Ht^2)\ddot{X} + (-t + C^2)\dot{X} = R_{\theta(t)}^{-1} A_i R_{\theta(t)} X \quad i = 1 \dots n$$

with A_i a 2×2 matrix (we can choose A_i diagonal after a basis change). Among the matrices A_i, there is one corresponding to the motion of the center of mass and this gives $A_1 = 0$. There is also a matrix corresponding to the Hamiltonian and total angular momentum (which are first integrals), and this corresponds to $A_2 = diag(2, -1)$. The other matrices do not have a priori special properties. Then the Galois group for the cases $i = 1, 2$ is $\mathbb{C}$, and for the others, it is at most $Sp(2)$. □

Proposition 14. *Let V be the potential of the n body problem with positive masses m_i, c an **aligned** Darboux point with multiplier -1 and $W \in M_{2n}(\mathbb{C})$ the associated matrix (given by equation (5.18)). The variational equation near a conic orbit is partially decoupled if and only if $det(W) = 0$.*

Proof. For an aligned Darboux point, we have the equalities (5.20). We define $v = (w_1, w_2)$. If v is an eigenvector of W, then w_1 is an eigenvector of A and $-\frac{1}{2}A$ with the same eigenvalue. Then $\det(A) = 0$. Conversely, if $\det(W) = 0$, then there exists an eigenvector w of eigenvalue 0 of A, and then $v = (w, w)$ is admissible. □

5.6.2 The 3 body problem and some specific cases

We already know that in all cases, the matrix W should have a double eigenvalue. Our approach will be the following. We search masses and Darboux points such that W has a double eigenvalue. Then for the corresponding eigenvector v, there are two possibilities.

- Either Jv is also an eigenvector of W with the same eigenvalue. This corresponds to the case where the associated eigenspace is of dimension ≥ 2.
- Or v can be written $v = (w, iw)$, and the matrix W is not diagonalizable.

For the aligned case, it is easier because we just have to look at the determinant. But in fact for the real ones, there is no zero eigenvalue if the Darboux point is real (this is due to the result of [59]), so we need to look at complex cases. But, even there, this constraint is much stronger than expected. We find the following theorem

Proposition 15. *Let V be the potential of the 3 body problem with positive masses m_1, m_2, m_3 such that $m_1+m_2+m_3=1$. Then V has a Darboux point such that the variational equation near a conic orbit is partially decoupled if and only if*

$$(m_1,m_2,m_3)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right),\left(\frac{1}{7},\frac{5}{7},\frac{1}{7}\right),$$
$$\left(\frac{1}{4}+\frac{\sqrt{21}+\sqrt{126+42\sqrt{21}}}{84},\frac{1}{2}-\frac{\sqrt{21}}{42},\frac{1}{4}+\frac{\sqrt{21}-\sqrt{126+42\sqrt{21}}}{84}\right) \tag{5.21}$$

or permutation of these cases.

Proof. Let us begin with aligned case. After renormalization, we can take $c=(-1,0,\rho)$ with $\rho\neq 0,-1$ and we have the Euler quintic equation

$$L=(-m_1-m_2)\,\rho^5+(-3m_1-2m_2)\,\rho^4+(-3\,m_1-m_2)\,\rho^3+$$
$$(3m_3+m_2)\,\rho^2+(3m_3+2m_2)\,\rho+m_2+m_3=0$$

We search the eigenvalues of W, and we find that $\det(W)=0$ if and only if

$$2\rho^2+3\rho+2=0$$

After taking the resultant, we have

$$\mathrm{Res}(2\rho^2+3\rho+2,L,\rho)=7m_2^2-35m_1m_2-35m_2m_3+56m_1^2+63m_1m_3+56m_3^2$$

We want this resultant to vanish, and the only possibility for real positive masses is

$$(m_1,m_2,m_3)=\left(\frac{1}{7},\frac{5}{7},\frac{1}{7}\right)$$

We can permute the masses in the equation and this gives all the possible permutations of this solution. But there is still a “complex order” and the corresponding potential is the following

$$V=\frac{m_1m_2}{q_1-q_2}-\frac{m_1m_3}{q_1-q_3}+\frac{m_2m_3}{q_2-q_3}$$

The Darboux point equation leads to

$$L=(-m_1-m_2)\,\rho^5+(-3\,m_1-2\,m_2)\,\rho^4+(-3\,m_1+2\,m_3-m_2)\,\rho^3+$$
$$(-2\,m_1+3\,m_3+m_2)\,\rho^2+(3\,m_3+2\,m_2)\,\rho+m_2+m_3=0$$

The eigenvalues of W never vanish in this case. Let us look now at the Lagrange configuration. For complex coordinates, this corresponds to the case

$$r_1^3=r_2^3=r_3^3$$

where r_1,r_2,r_3 are the mutual distances between the bodies. We begin by the case $r_1=r_2=r_3$. We need a double eigenvalue and we find the condition

$$3m_2^2-3m_2m_3-3m_1m_2+3m_3^2-3m_1m_3+3m_1^2=0$$

whose unique solution is

$$(m_1,m_2,m_3)=\left(\frac{1}{3},\frac{1}{3},\frac{1}{3}\right)$$

We check that the associated eigenspace of eigenvalue $1/2$ is invariant by J, and it is the case. Let us look now at the complex cases. Among the $27-1$ possibilities lots of them are in fact the same after dilatation-permutation. After these reductions, we find that there are only 3 essentially different cases

$$(r_1,r_2,r_3)=(1,1,j),(1,1,j^2),(1,j,j^2) \qquad j=e^{\frac{2i\pi}{3}}$$

The last one is also an aligned Darboux point (it is both Lagrange and Euler configuration), therefore it has already been treated. First we search for masses such that W has a double eigenvalue. We find for $(1,1,j)$ and $(1,1,j^2)$ a single real positive solution, which is the last one of (5.21). This is the same for both Darboux points because they are conjugated. We look at the corresponding eigenspace (the double eigenvalue is $1/2$), and we find that the eigenspace is only 1-dimensional. This is not enough for the case $\dim(\tilde{V})\geq 2$. In the case $\dim(\tilde{V})=1$, we know that W should be non-diagonalizable. Moreover, the eigenvector should be of the form $v=(w,iw)$. We check these properties and they are satisfied. □

Remark 20. *The last case of* (5.21) *is very interesting for many reasons. We can study the variational equations and we see that the structure of the equations is not so degenerate as in the other cases. Because of this, a far deeper analysis should be possible. For example, another notion of partial integrability is considered in [55, 48], about the existence of a single additional first integral. For this last masses case, the two notions could probably be fused together to prove the non existence of a single additional first integral restricted to a single level of the Hamiltonian and total angular momentum. This is because the variational equation on the characteristic space associated to the eigenvalue* $1/2$ *is simple enough to allow complete study, but is not trivial. Moreover, the fact that these masses do not admit any symmetry will avoid to consider special invariant submanifold as the isosceles* 3 *body problem in, for example, the complete search of algebraic invariant manifold for the* 3 *body problem with these masses.*

Proposition 16. *(See [24] for examples) We consider V the potential of the n body problem in the plane with positive masses, and c a real central configuration such that there exists a rotation*

$$R_\theta,\;\; \theta\notin\{k\pi,\; k\in\mathbb{Z}\}$$

in the plane which sends the configuration on itself (conserving also the masses). Then there exists a double eigenvalue and the associated eigenspace is of dimension ≥ 2.

Proof. Let R_θ be a rotation such that $\theta\notin\{k\pi,\; k\in\mathbb{Z}\}$ which sends the configuration c on itself and conserves the masses. We consider the matrix $W(c)$ as in (5.18) and we have then the identities

$$W(R_\theta c)=R_{-\theta}W(c)R_\theta \qquad W(R_\theta c)=W(Pc)=P^{-1}W(c)P$$

with P a permutation matrix (the rotation conserves the configuration and the masses of the bodies, but not the numeration of the bodies). Then

$$W(c)=(R_\theta P^{-1})^{-1}W(c)R_\theta P^{-1}$$

Let v be an eigenvector of $W(c)$. Then $R_\theta P^{-1}v$ is also an eigenvector with the same eigenvalue. We just have to prove it is not the same. We can write in a good basis

$$R_\theta=\begin{pmatrix}\cos\theta I_n & -\sin\theta I_n\\ \sin\theta I_n & \cos\theta I_n\end{pmatrix} \qquad P=\begin{pmatrix}P_\sigma & 0\\ 0 & P_\sigma\end{pmatrix}$$

with P_σ a permutation matrix. We find then that P and R_θ commute. We know that the rotation R_θ is of finite order (because there are only a finite number of bodies and that the configuration is real). Then $\theta=2\pi/k$ with $k\in\mathbb{N}^*$, and $k\geq 3$. The matrix P is then also of order k.

Let us consider the body number i with coordinates q_i. We look at the orbit ${R_\theta}^j q_i,\; j=0...k-1$. This orbit contains either k elements or a single one (and this case could only happen once, for a body placed on the center of mass). We conclude that the permutation matrix should be of the following form

$$P_\sigma = \begin{pmatrix} T & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & T & 0 \\ 0 & \dots & 0 & Id \end{pmatrix} \text{ or } P_\sigma = \begin{pmatrix} T & 0 & \dots & 0 \\ 0 & \dots & 0 & 0 \\ 0 & \dots & T & 0 \\ 0 & \dots & 0 & T \end{pmatrix} \text{ with } T = \begin{pmatrix} 0 & 1 & \dots & 0 \\ 0 & 0 & 1 & 0 \\ 0 & \dots & 0 & 1 \\ 1 & 0 & \dots & 0 \end{pmatrix}$$

We conclude that the matrix $R_\theta P$ can be diagonalized in the form

$$R_\theta P \sim \text{diag}\left(e^{i\theta}, e^{-i\theta}, \left(e^{i(j+1)\theta}, \dots, e^{i(j+1)\theta}, e^{i(j-1)\theta}, \dots, e^{i(j-1)\theta}\right)_{j=0..k-1}\right)$$
$$\text{or} \quad R_\theta P \sim \text{diag}\left(\left(e^{i(j+1)\theta}, \dots, e^{i(j+1)\theta}, e^{i(j-1)\theta}, \dots, e^{i(j-1)\theta}\right)_{j=0..k-1}\right)$$

We know that the masses are positive and that the Darboux point is real, then all eigenvectors v of $W(c)$ are real. Assume that $W(c)$ does not have any eigenspace of dimension ≥ 2. Then all its eigenvectors are eigenvectors of $R_\theta P$. As $R_\theta P$ is real, if v is a real eigenvector of $R_\theta P$, then the associated eigenvalue is real and so the associated eigenvalue is ± 1. This would mean that

$$\text{Sp}(R_\theta P) \subset \{-1, 1\}$$

This is impossible because $k \geq 3$. □

Remark that the vector space E generated by $(R_\theta P^{-1})^k v,\ k \in \mathbb{Z}$ is a subspace of the eigenspace and is stable by J. So thanks to Lemma 28, the variational equation near the conic orbit associated to c is decoupled, and we can use Theorem 35.

5.6.3 The equal masses case

Lemma 29. *Let V be the potential of the n body problem in the plane with equal masses, c the Darboux point given by the following*

$$c_i = \alpha cos\left(\frac{2\pi(i-1)}{n}\right) \qquad c_{i+n} = \alpha sin\left(\frac{2\pi(i-1)}{n}\right) \qquad i = 1 \dots n$$

where α is such that the multiplier equals to -1. Let v be the vector given by

$$v_i = \cos\left(\frac{4\pi(i-1)}{n}\right) \qquad v_{i+n} = \sin\left(\frac{4\pi(i-1)}{n}\right) \qquad i = 1 \dots n$$

Then (5.19) *is satisfied with*

$$\lambda = 2 - \frac{2\sin\left(\frac{\pi}{n}\right)}{1-\cos\left(\frac{\pi}{n}\right)} \left(\sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{\pi j}{n}\right)}\right)^{-1} \tag{5.22}$$

Proof. The proof is only a direct computation of the matrix W and then of Wv and the use of (lots of) trigonometric formulas.

□

We can now eventually prove Theorem 32.

Proof. Using Lemma 27, one just need to avoid specific values for λ. We will then build a majoration and minoration for λ given by formula (5.22). First of all, we remark that for $n \geq 3$

$$\frac{2\sin\left(\frac{\pi}{n}\right)}{1-\cos\left(\frac{\pi}{n}\right)} \left(\sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{\pi j}{n}\right)}\right)^{-1} > 0$$

Then $\lambda < 2$. Let us prove now that $\lambda > 0$. First we prove the following inequality

$$\sin(z) < \frac{1}{\sin(z)} \quad \forall z \in]0, \pi/2[\cup]\pi/2, \pi[$$

and we compute the formula

$$\frac{\sin\left(\frac{\pi}{n}\right)}{1 - cos\left(\frac{\pi}{n}\right)} = \sum_{j=1}^{n-1} \sin\left(\frac{\pi j}{n}\right)$$

Using both of them, this gives for $n \geq 3$

$$\frac{\sin\left(\frac{\pi}{n}\right)}{1 - \cos\left(\frac{\pi}{n}\right)} < \sum_{j=1}^{n-1} \frac{1}{\sin\left(\frac{\pi j}{n}\right)}$$

So we get that $\lambda > 0$. Using the integrability table of Lemma 27, there are no exceptional values in $]0, 2[$. □

The case $C^2 H = 0$ is special. In this case, we have either $C = 0$ or $H = 0$ (or both). The case $H = 0$ corresponds to parabolic orbits. These orbits are used by Tsygvintsev in [68], and he solves the case for 3 bodies with equal masses (he also studies the existence of a single additional first integral in [69] that we do not consider). In the case of $C = 0$, the problem is solved by Morales, Simon in [55] for the n equal masses. Our reasoning is also valid for all these cases.

Remark 21. *The only left case is $H = C = 0$. Here, the variational equation is always integrable and in fact it is always the case at all orders. This is linked to the fact that we can reduce the system using homogeneity and rotation, allowing to diminish the dimension of* 4*. We obtain then a "direction" field (we loose notion of time after reduction, but not the integrability notions) on a manifold of dimension $4n - 8$. This, however, destroy Hamiltonian structure, and moreover, the Darboux points correspond now to fixed points of this field. One would need a new particular orbit (explicit) to apply Morales Ramis method, but no such orbit is known.*

Conclusion et perspectives

Nous avons ainsi démontré de nombreux critères d'intégrabilité, assez forts pour aborder la plupart des problèmes de non-intégrabilité de potentiels homogènes de degré -1 sur des familles avec un nombre fini de paramètres. Les problèmes en grande dimension peuvent cependant encore être difficiles, et cela à cause des obstacles suivants

- Le calcul de la majoration des valeurs propres Λ elle même peut être très longue et difficile, et on peut penser que cette borne si elle existe aura tendance à augmenter très rapidement avec la dimension
- Dans le cas où les valeurs propres sont bornées, le nombre de combinaisons de valeurs propres peut devenir très grand même si la borne est petite, par exemple ne serait ce qu'en dimension 5 avec $\Lambda < 100$, on arrive déjà à plusieurs dizaines de milliers de cas, alors que chaque cas peut prendre de nombreuses heures à être résolu (c'est à dire arriver à montrer qu'ils sont impossibles)
- Enfin, le cas des familles avec $\Lambda = \infty$ ne peuvent pour l'instant être traitées qu'en dimension 2, sauf dans certains cas favorables. En effet, le calcul de la condition d'ordre 3 en n'importe quelle dimension (un petit raisonnement montre qu'il suffirait de le faire en dimension 4) semble pour l'instant hors de portée du point de vue cout de calcul. En effet, la manipulation d'expression D-finies se révèle, bien que théoriquement toujours possible, très couteuse.

Au delà de ces considérations pratiques, le but ultime serait de pouvoir trouver tous les potentiels méromorphes intégrables homogènes de degré -1, et ce en toute dimension. On a vu d'après les divers résultats précédents que le fait de supposer que le potentiel soit réel simplifie beaucoup le problème. C'est très étonnant à première vue car rien dans le théorème de Morales Ramis n'impose une telle condition. Tout vient du théorème du maximum, qui a une importance capitale dans les questions de non-intégrabilité. En effet, un point de Darboux peut être vu comme un point critique du potentiel restreint à la sphère unité, et ainsi nous garantit l'existence d'un tel point dans de nombreux cas. Son impact est encore plus fort en dimension 3 et plus car les conditions d'intégrabilité deviennent si compliquées et restrictives qu'elles ne peuvent pas être vérifiées sur des exemples concrets (même avec l'ordinateur), mais il est cependant possible de montrer qu'elles n'ont pas de solutions réelles. Ainsi, on peut faire les conjectures suivantes

Conjecture 3. *Soit V un potentiel homogène méromorphe de degré -1 en dimension n, réél et tel que la restriction de V à la sphère unité est bornée. Si V est méromorphiquement intégrable, alors*

$$V(q) = \frac{a}{\sqrt{\sum_{i=1}^{n} q_i^2}} \qquad a \in \mathbb{R}$$

Soit V un potentiel homogène méromorphe de degré -1 en dimension 3 réél et tel qu'il existe une rotation d'ordre ≥ 4 qui laisse invariant V avec $V \neq 0, \infty$ sur l'axe de rotation. Si V est *méromorphiquement intégrable, alors*

$$V(q) = \frac{a}{\sqrt{q_1^2 + q_2^2 + q_3^2}} \qquad a \in \mathbb{R}$$

Cette conjecture semble accessible car soit on est à valeur propres bornées ($\Lambda = -1$), soit on est non borné ($\Lambda = \infty$) mais les calculs semblent d'une difficulté raisonnable. On peut aussi conjecturer quelques résultats très généraux bien que sans doute inaccessibles pour l'instant.

Conjecture 4. *On pose $\mathcal{M}$ l'ensemble des potentiels homogènes méromorphes de degré -1 en dimension n. Soit $V \in \mathcal{M}$, on note $d(V)$ l'ensemble des points de Darboux c de V tels que $V'(c) = -c$ et V holomorphe en c. Pour $c \in d(V)$ on a $Sp\left(\nabla^2 V(c)\right) = \{2, \lambda_1, \dots, \lambda_{n-1}\}$ et on note*

$$\Lambda_{int}(\mathcal{M}) = \sup_{V\in\mathcal{M}, d(V)\neq\varnothing} \inf_{c\in d(V)} \begin{cases} \max(\lambda_1,\dots,\lambda_{n-1}) \quad si \\ V \textit{méromorphiquement intégrable} \\ -\infty \quad sinon \end{cases}$$

Alors $\Lambda_{int}(\mathcal{M}) = 2$.

Avec ce type de conjecture, on aurait alors une bonne majoration sur les valeurs propres aux points de Darboux. Cependant cette conjecture semble pour l'instant inabordable, et ce même en dimension 2. Les techniques de création télescopiques devraient être capable de la démontrer, mais les calculs sont inabordables (et de loin). En effet, il suffirait de calculer les équations variationnelles à l'ordre 7, exprimer les contraintes de monodromie comme des intégrales de chemin puis construire une suite récurrente pour ces éléments de monodromie. La question de la dimension quelconque ne pose pas de problème car pour les équations variationnelles à l'ordre 7, même en dimension d quelconque, on peut se ramener à des équations qui ne font intervenir qu'un nombre fini de dérivées de V, et ce indépendamment de d. Cela permettrait alors de vérifier que la condition d'intégrabilité correspondante ne peut pas être vérifiée pour des valeurs propres supérieures à 2. En pratique, cela correspondrait à une suite récurrente en dimension 8, dont les degrés des coefficients seraient inimaginables. Plus raisonnablement, avec des progrès substantiels sur les algorithmes de manipulations des fonctions D-finies et meilleure compréhension des particularités du problème, il devrait être possible de résoudre la question en dimension 2. Cela permettrait d'enlever le mot "générique" sur la classification des potentiels homogènes intégrables méromorphes du plan de degré -1. La classification complète en toute dimension serait alors probablement

Conjecture 5. *Soit V un potentiel méromorphiquement intégrable de $\mathcal{M}$ qui possède au moins un point de Darboux non dégénéré. A rotation-symétrie près, V est égal à*

$$V = \sum_{j=1}^{k} \frac{a_j q_{2j}}{(q_{2j} + \epsilon_j i q_{2j-1})^2} + \sum_{l=1}^{m-1} b_l \left(\sum_{j=j_l+1}^{j_{l+1}} q_j^2 \right)^{-1/2} \tag{5.23}$$

avec $2k = j_1 < j_2 < \dots < j_{m-1} < j_m = n$ et $a_j, b_l \in \mathbb{C}, \epsilon_j \in \{-1, 1\}$.

Cette conjecture est sans doute abordable si l'on admet la première. D'ailleurs, d'un point de vue général, même en dimension quelconque (finie et fixée), une famille de potentiels dont les valeurs propres sont bornées devrait pouvoir être étudiée du point de vue de l'intégrabilité, avec les mêmes méthodes que nous avons présenté. En effet, l'algorithme utilisé pour démontrer qu'un certain ensemble de valeurs propres n'est pas permis pour l'intégrabilité à l'ordre 7 devrait pouvoir être généraliser à n'importe quelle dimension. Ce nous donnerait de plus un algorithme qui, pour tout potentiel homogène rationnel donné et un point de Darboux, nous renvoie à quel ordre (≤ 7) il est intégrable.

De plus, la plupart des résultats de cette thèse peuvent être généralisés à n'importe quel degré d'homogénéité $\neq -2, 0, 2$, avec cependant sans doute beaucoup d'efforts pour les degrés

$-5, -4, -3, 3, 4, 5$ qui présentent des difficultés particulières. Il s'agit en effet essentiellement de remplacer les fonctions $\text{arctanh}(1/t)$ par des fonctions hyperelliptiques. Du point de vue algorithmique, les résultats précédents nous montrent d'ailleurs que les théorèmes disponibles en non-intégrabilité sont suffisamment avancés et contraignants pour avoir une approche algorithmique de la non-intégrabilité. Il s'agirait alors de construire deux algorithmes tels que

- Etant donné un potentiel homogène rationnel V à coefficients dans $\mathbb{Q}$, un point de Darboux c et un entier naturel n, l'algorithme nous dit si le potentiel V est intégrable ou non à l'ordre n au voisinage de c
- Etant donné une famille de potentiels homogènes rationnels V à coefficients dans $\mathbb{Q}$ paramétrés rationnellement avec un nombre fini de paramètres et un entier naturel n, l'algorithme nous renvoie toutes les valeurs des paramètres tels que le potentiel correspondant soit intégrable à l'ordre n au voisinage de tous les points de Darboux

Le premier problème est décidable complètement, mais en pratique, il est préférable d'utiliser une méthode heuristique pour chercher des contraintes d'intégrabilité pour gagner en vitesse d'exécution. Cependant, comme nous l'avons dit tout au début de cette thèse, ce problème en lui même n'est pas très intéressant, car montrer qu'un potentiel particulier n'est pas intégrable n'avance en général pas à grand chose. La seconde question est bien plus intéressante, mais la question de la décidabilité reste ouverte. En effet, on a la relation de Maciejewski-Przybylska

$$\sum_{i=1}^{n} \frac{1}{\lambda_i + a} = b \tag{5.24}$$

valable génériquement, mais pas valable dans tous les cas. En effet, une des conditions est que tous les points de Darboux soient simples. Ce cas particulier peut cependant être contourné par une méthode similaire à celle du chapitre 3 car une des valeurs propres de la matrice hessienne est alors fixée. Mais il existe une autre condition, qui n'est par exemple pas satisfaite par la famille de potentiels étudiée au chapitre 4, et qui correspond au cas où les valeurs propres sont non bornées. L'analyse à l'ordre 3 en utilisant les méthodes de création télescopique semble alors inévitable, et l'on peut penser ainsi que la question est décidable pour $n \geq 3$ (tout du moins en dimension 2) mais pas pour $n = 1, 2$. En effet, si l'on a pas la relation de Maciejewski-Przybylska, il semble que l'on puisse tomber sur n'importe quelle équation Diophantienne, problème qui est évité à l'ordre 3 et plus car ayant une condition d'intégrabilité supplémentaire (ce qui permet de borner les entiers de l'équation Diophantienne précédente). Remarquons que le cas particulier du degré d'homogénéité impair en dimension 2 évite ce problème, et la question est alors décidable.

Ce type d'approche permet de résoudre de grand nombre de systèmes avec beaucoup de paramètres, mais avec cependant la condition que le potentiel V ne soit pas trop compliqué. En effet, on peut craindre même dans le cas générique une complexité au moins en $O(N!)$, ce qui rend par exemple cette approche infaisable dans le cas de problèmes de n corps (mais en s'en doutait déjà car cela implique le calcul de toutes les configurations centrales).

Du point de vue de la mécanique céleste, il devrait être possible de montrer la non-intégrabilité de 5 corps alignés, mais guère plus, sauf pour des masses particulières. Cependant de tels résultats deviennent alors de toute façon plus des démonstrations calculatoires que des résultats intéressants mathématiquement. Les problèmes intéressants sont à mon avis ailleurs, et notamment le cas du moment cinétique non nul. En effet, le théorème 13 nous dit que l'équation variationelle au voisinage d'une orbité Képlérienne associée à une configuration centrale alignée peut s'écrire sous la forme d'équation différentielles $\dot{X} = A(t)X$ avec $A \in M_4(\mathbb{C}(t))$. Ainsi, un algorithme de type Kovacic pourrait calculer le groupe de Galois de cette equation variationelle s'il existait en dimension 4. Il n'existe pour l'instant que jusqu'à l'ordre 3. On peut tout de même espérer que l'équation variationnelle possède suffisamment de singularités logarithmiques pour exclure d'office les groupes de Galois les plus compliqués, et ainsi rendre la tâche faisable [17].

Enfin, et c'est sans doute le plus intéressant, essayer de trouver des sous variétés du problème de n corps qui soient invariantes et complètement intégrables. En particulier, on peut généraliser la notion de configuration centrale (et de point de Darboux) par la notion suivante

Définition 5. *Soit V un potentiel homogène en dimension n. On dira qu'un sous espace vectoriel W de $\mathbb{C}^n$ est un sous espace stable de V si pour presque tout $(q_1,\dots,q_n)\in W$ (c'est à dire partout sauf sur éventuellement sur une sous variété de W de codimension au moins 1), on a*

$$V'(q_1,\dots,q_n)\in W$$

C'est une (parmi sans doute d'autres) possibles généralisation des points de Darboux. En effet, un point de Darboux correspond au cas $dim\ W=1$, et dans le cas des configurations centrales du problème de n corps en dimension d, cela correspond à $dim\ W\leq d$ car le potentiel est toujours invariant par rotation. Les cas de dimension plus grande correspondent souvent à ses sous systèmes particuliers connus, comme le problème isocèle et plus généralement des problèmes avec des symétries. Leur classification étant plus difficile que le problème des configurations centrales, on ne peut sans doute pas en espérer une description complète. Cependant, on sait que la restriction a un tel sous espace va encore donner un potentiel homogène de degré -1 en $dim\ W$ variables. La condition de majoration des valeurs propres s'il reste encore des paramètres reste souvent vraie, mais surtout il se pourrait que cela donne un système intégrable. Pour "intuiter" de tels systèmes, on peut par exemple admettre la conjecture 5 et s'en servir comme guide pour trouver de tels systèmes.

On a alors la "règle" suivante. Pour qu'un problème de n corps en dimension d soit complètement intégrable sur un sous espace vectoriel stable W, il faut que le potentiel V restreint à W se mette sous la forme

$$V=\sum_{j=1}^{k}\left(\sum_{i=1}^{i_j}(L_{i,j}(q))^2\right)^{-1/2}\qquad \sum_{j=1}^{k} i_j\leq dim\ W$$

avec $L_{i,j}$ des formes linéaires en q. Il suffit pour démontrer cette condition nécessaire de regarder les potentiels (5.23) et de voir que le nombre de termes dans la somme reste inchangé après une rotation quelconque ainsi que cette structure en somme de termes quadratiques et le rang de ces formes quadratiques.

On trouve alors les cas suivants

- Le problème de 5 corps avec des masses $1,1,1,1,-1/4$ de coordonnées $q_1,\dots,q_5\in\mathbb{C}^2$ et les conditions $q_5=0, q_1+q_3=0, q_2+q_4=0$. Ce sous espace contient en particulier le continuum de configuration centrales trouvé par Roberts [64]. L'espace vectoriel W est de dimension 4, et le potentiel V_1 correspondant est complètement intégrable. Le même système peut aussi être construit en dimension 3, ce qui donne un espace vectoriel W de dimension 6 complètement intégrable.
- Le problème de $n+3$ corps avec des masses $m,m,-1/4m,1,\dots,1$ avec

  $$m=2\sum_{k=1}^{n-1}\sin\left(\frac{k\pi}{n}\right)^{-1}$$

  et sous les conditions que les n dernières masses forment un polygone régulier à n coté de centre 0, la masse $-1/4m$ est placée en 0 et les deux masses m sont placées symétriquement par rapport à 0 sur l'axe orthogonal au plan du polygone régulier. L'espace vectoriel est de dimension 3, et le potentiel V_2 correspondant est complètement intégrable.
- Plus généralement, le problème de $n+3$ corps consistant en n corps en configuration centrale cocyclique, un corps de masse négative m bien ajusté, et deux autres corps symétriquement sur la droite normale au plan de la configuration centrale de masse $-4m$.

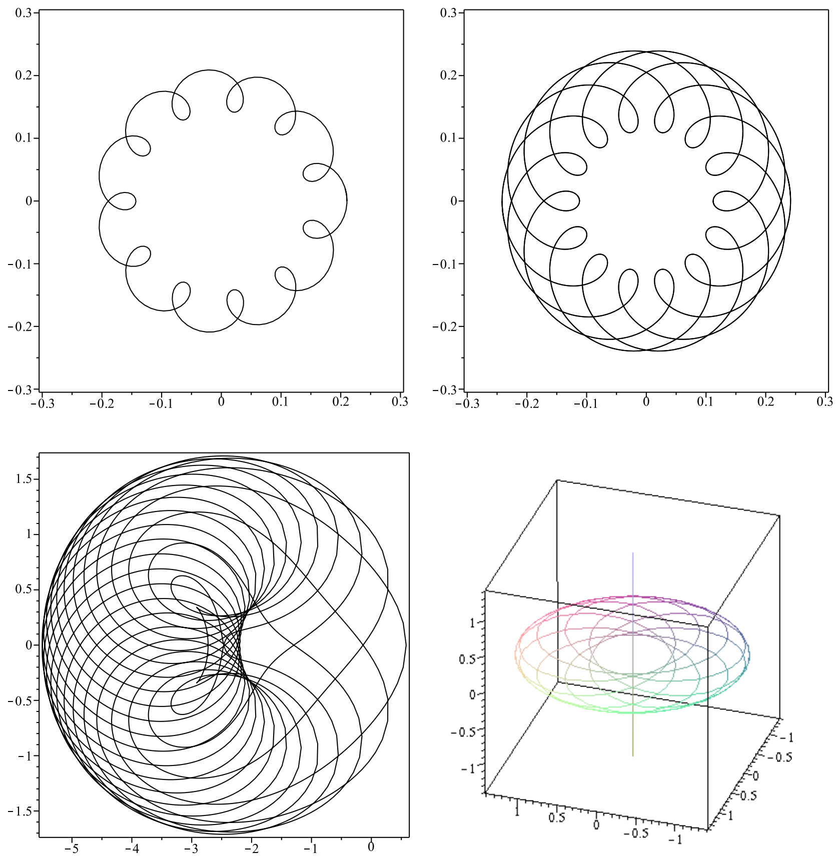


Figure 5.1: Les trois premiers graphiques representent le mouvement d'un des corps de masse 1 pour le système intégrable à 5 corps dans le plan. Le premier est à excentricité nulle, le deuxième à ecentricité non nulle mais alignement des périhélies et le troisième est arbitraire, avec cependant dans les 3 cas un rapport de période rationel pour les deux mouvements elliptiques. La dernière figure représente le mouvement de $n+3$ corps avec ellipticité non nulle.

Qu'il existe de tels sous systèmes non trivialement intégrables dans le problème de n corps semble très surprenant tant ce problème a déjà été étudié. Le mouvement de V_2 correspond à un mouvement képlerien en dimension 3. Toutes les orbites sont donc périodiques, et les trajectoires des corps sont sur des ellipses (mais pas les mêmes ellipses pour tous les corps!). Le mouvement de V_1 est lui particulièrement intéressant et complexe en apparence. Il se réduit à deux mouvements képleriens en dimension 2 indépendants après un changement de variable. Plus précisément, les 4 corps de masses 1 forment un parallélogramme à tout temps dont le milieu est 0, et les milieux des cotés de ce parallélogramme suivent des trajectoires képleriennes indépendantes. Cela produit des orbites périodiques ou pseudo-périodiques en fonction du rapport des fréquences des deux mouvements képleriens, des chorégraphies algébriques pour les 4 corps de masses 1. Il semble y avoir un lien entre l'existence de tels sous espaces vectoriels et la question de la finitude du nombre de configurations centrales, car ces espaces vectoriels possèdent en particulier des continuums de configurations centrales. On remarque qu'on l'on a eu besoin ici de masses négatives, mais rien n'interdit a priori qu'il y ait des cas avec des masses toutes positives, cependant il est conjecturé que le nombre de configurations centrales en masses positives est fini, et on peut donc conjecturer que là aussi imposer que les masses soient toutes positives interdira ces cas particuliers.

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